

MONTHLY NOTICES
OF THE
ROYAL ASTRONOMICAL SOCIETY

Vol. 94

No. 2

MEETING OF 1933 DECEMBER 8

Prof. F. J. M. Stratton, President, in the Chair.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended :—

Raymond Arthur Lyttleton, B.A., Clare College, Cambridge (proposed by W. M. Smart) ;

Vincent Oswald Woodfield Peters, Lieut., R.N.R., 36 Connought Road, Hong Kong, China (proposed by P. F. Everitt) ;

Somasundram Sabapathipillai, 112 Gower Street, London, W.C. 1 (proposed by C. C. L. Gregory) ;

Arthur Geoffrey Walker, B.A., Ph.D., Merton College, Oxford (proposed by E. A. Milne) ; and

Donald Arthur Wright, B.Sc., 154 Bournbrook Road, Selly Oak, Birmingham (proposed by M. C. Johnson).

Sixty presents were announced as having been received since the last meeting, including :—

Thirty-five lantern slides of the Adler Planetarium and Astronomical Museum, Chicago (presented by Dr. P. Fox) ;

P. Fox, "The Adler Planetarium and Astronomical Museum: An Account of the Optical Planetarium and a Brief Guide to the Museum" ;

Campidoglio Observatory, Rome, "Catalogo di Stelle compilato sulle Osservazioni . . . da A. di Legge, F. Giacomelli, 1911" ;

P. Stroobant, "Précis d'Astronomie."

Mr. C. O. Bartrum and Mr. D. H. Sadler were appointed auditors of the Treasurer's accounts for 1933.

ON THE DETECTION OF SMALL DOPPLER SHIFTS IN THE SPECTRUM OF THE REVERSING LAYER.

J. Evershed, F.R.S.

The positive on negative method of measuring small shifts in spectrum lines, first described in 1913 in *Kodaikanal Bulletin*, No. 32, has been applied recently to some of Dr. Hale's original spectra upon which he based his discovery of a general magnetic field in the Sun. In these spectra the Zeemann effect is much too small to show the separation of lines into doublets or triplets, such as are seen in the much stronger fields in sunspots. The lines are very slightly widened, and by means of polarizing apparatus placed in front of the slit of the spectrograph the red and violet components of the Zeemann doublets are cut out alternately in the contiguous narrow strips into which the spectrum is divided. This has the effect of shifting the centre of the line slightly towards red in a strip and slightly towards violet in the adjoining strip, the whole length of a spectrum line representing a section of the Sun's image in varying latitudes along the central meridian.

The method I adopted in measuring these plates was to take an ordinary contact positive and place it film to film over the negative in the micrometer, without reversing the plates end for end. Each line in the negative was therefore covered by the corresponding line in the positive. Then, by shifting the positive laterally by one 2-mm. interval, the positive images having red shifts were superposed on the negative images having violet shifts, and *vice versa*. In this way, bringing the positive and negative images in each strip in succession into exact coincidence by moving the negative with the micrometer screw, the positive being fixed in position above it, the difference of readings gave twice the actual shift between successive strips.

The method seems to be superior in three ways to other methods of measuring such spectra. First, the interval measured is twice the actual shift; secondly, the adjustment for coincidence can be made with greater accuracy than in other methods; and thirdly, the width of a line has little effect on the accuracy of adjustment. This last is of value in measuring plates having a great linear dispersion, in which even the narrowest solar lines have very appreciable widths. The grating plates of the magnetic field have a dispersion of 5 mm. to the angstrom unit, and shifts of the order of $\cdot 001$ A. can be measured with certainty, since this change of wavelength is represented by a movement of the micrometer screw of $\cdot 01$ mm.

The general result of the few plates measured was very satisfactory. The right order of shift was obtained and the right sign in the two hemispheres of the Sun, but, as in all spectrum measurements, anomalies were met with in most of the plates, and especially in low latitudes where the Zeemann effect would be smaller. These might be due to local magnetic fields, or to the disturbing effects of small Doppler shifts. The plates

were obtained at a time of minimum sunspots, when disturbances giving motions in the line of sight might be expected to be small, or even non-existent.

In view of these results I have now made a special effort to discover whether such small Doppler shifts do occur at the present time of minimum solar activity. I had already found irregular movements extending over large areas in the immediate neighbourhood of sunspots, and these are very easily detected by the positive on negative method. In this case it is necessary to obtain a positive by photographing the negative through the glass in a camera. The negative is placed film outwards, and the camera lens arranged exactly midway between the film and the plate. This gives a positive image the same size as the negative, and is equivalent to copying through the glass instead of film to film. If this positive is placed film to film on the negative, but reversed end for end, the positive image of a line may be superposed on the negative image of the same line, and the condition is fulfilled whereby any point along the negative image coincides with the same point on the positive image. If there is a change of wave-length at this point, the displacement will be in opposite directions in the two images. This has the effect of making small shifts plainly visible, and if the positive is made on a slow plate of good contrast and density the line will appear dark, but no longer straight, in fact near to sunspots the finer lines often show marked displacements to red and violet alternately, taking a sinuous course.

I have adopted this procedure to detect and measure small Doppler effects which may occur at the present time of minimum activity. High-dispersion spectra of sections of the Sun's disc have been obtained with four transmissions through the liquid prism spectrograph. These have a linear dispersion equal to 1.8 mm. to the angstrom at $H\beta$, increasing to 2.8 mm. at $H\gamma$. The slit is 20 mm. long, and covers one-third of the diameter of the Sun's image. Exposures of from 30 seconds to 4 minutes are required, and are made only under good conditions of seeing: with bad seeing, small localized disturbances are more or less reduced or entirely obliterated. A new slit curved to a radius of 11 inches had to be made in order that the spectrum lines should be perfectly straight. Otherwise the coincidence of positive and negative images when reversed could not be made along the whole length of a line.

Up to the present, 24 plates have been secured of sections of the disc between the centre and the limb, and in all of these small shifts and sinusities are seen in the narrower lines—that is to say, lines of Rowland's intensities 2, 3 and 4. The displacements of course vary from plate to plate, showing that they are not due to irregularities in the slit of the spectrograph. In any one plate they occur at the same points in all the lines examined: they cannot therefore be ascribed to accidental irregularities in the silver deposit. Ten plates have been taken also with the slit crossing the centre of the disc, and in eight of these the shifts appear to be entirely absent. On two, however, very small irregularities are shown either at the centre or near to it.

The measured shifts in the region between $H\beta$ and $H\gamma$ range from .002 to .005 A., the majority of those easily visible ranging between .003 and .004 A. Shifts smaller than .002 A. were not estimated.

It is evident from these preliminary results that measures of the Zeemann effect in the general magnetic field, which involve shifts of the order of .001 A. only, may be largely affected by what may be considered accidental shifts of considerably greater magnitude. The effects of these disturbances are, however, minimised by the very long exposure time needed in photographing the high-dispersion grating spectra, whereby the inevitable movements of the Sun's image on the slit will tend to obliterate any localised Doppler effects without influencing the general Zeemann effect. Also as the positive image in measuring these plates is not copied through the glass or reversed on the negative, but is displaced laterally by an amount corresponding with the change of sign of the Zeemann shifts, the local Doppler shifts will tend to be reduced in amount, whilst the Zeemann shift is always multiplied by two in the measures.

Assuming the shifts I have observed to be Doppler effects, the fact that they appear mostly at a distance from the centre of the disc, and rarely at the centre, would indicate that the movements causing them are mainly parallel to the solar surface. In this they resemble the radial motion in sunspots, in which vertical motion is rarely detectable. The motion outwards from the umbra being horizontal, the component in the line of sight vanishes when the spot is near the centre of the disc.

The actual motion indicated by the measures cannot be computed exactly. The mean value of the component in the line of sight from 36 measures is .21 km./sec., which becomes .36 km./sec. if multiplied by the cosecant of the mean angular distance from the centre, but this assumes the motion to be in the direction of the centre of the disc. The actual motion would in general be greater: it is in any case rather surprisingly large.

Experience so far indicates that these movements are extremely local. They cannot therefore be considered as winds in the solar atmosphere extending over appreciable areas: more probably they may be thought of as incipient sunspot disturbances. They are apparently confined to the lower reversing layer, and are therefore not easily detected near the Sun's limb where the lines represent higher levels.

It is hoped to determine the latitudes in which shifts are most frequent, and possibly to correlate them with the beginning of the new sunspot cycle.

ANOMALOUS MULTIPLET INTENSITIES IN STELLAR SPECTRA AND SUNSPOTS.

A. D. Thackeray, B.A.

A spectrophotometric study of thirty late-type stars has been made in the green region, measurements being restricted to equivalent breadths of the D lines of sodium, the magnesium " b " triplet and the $a^5F - z^5D^0$ multiplet of iron consisting of twelve lines (5269–5506). The present paper deals only with the results of the multiplet, which are compared with similar results for other multiplets in the solar and sunspot spectrum. The behaviour of sodium and magnesium will be considered in a later paper.

§ 1. *Method.*—The plates were all taken with the Huggins 15-inch refractor and composite prism spectrograph giving a dispersion of 50 Å./mm. at λ 5183, and standardised in a separate spectrograph with a neutral-tinted wedge placed in front of the slit. The plates usually used were Ilford Special Rapid Panchromatic, but on bright stars Process plates were used to gain the advantage of their remarkably fine grain, and for some of the fainter stars Ilford Hypersensitive Panchromatic plates were used. The plates were brush-developed in Metol-quinol developer and tracings taken with the photo-electric microphotometer designed by Carroll and Moss.* Characteristic curves were obtained for each plate at three wave-lengths (5000, 5400, 5800 Å.); it was noted that the curves were sensibly the same at the latter two wave-lengths, but at 5000 (near the minimum sensitivity of the plates) the contrast γ was slightly smaller, though the effect could be diminished by gaining contrast with increased development. The tracings were reduced by drawing in the background of continuous spectrum, and finding points corresponding to 90, 80, 70 per cent., etc., of this light; the breadths of each line were then measured at each of these points and the total intensity deduced (in angstroms and units of complete absorption). The lines were usually sufficiently narrow to take the contour as linear between each measured breadth; thus the total area was obtained by summing a series of trapezia and a triangle at the lowest point.

It was found convenient to develop together pairs of plates (cut from the same original plate); thus a single characteristic curve would be deduced for the two plates from two separate measurements, and some accidental errors of reduction be diminished. All stars except those likely to be of little interest were photographed at least twice, several of the brightest stars up to five times. Exposures ranged from 15 minutes to two and a half hours for the faintest stars (3rd mag.).

§ 2. *Errors.*—The numerous errors, accidental and systematic, that such work is liable to may be divided into three main groups—(a) errors in

* *M.N.*, 91, 191, 1931.

obtaining the spectrum, including both instrumental errors of the spectrograph and photographic errors; (b) errors in obtaining the microphotometer tracings; (c) errors in reducing the tracings. Errors (b) and (c) can be lessened by taking duplicate tracings and reductions; errors (a) are permanently present in any one plate, and can only be dealt with by taking several plates on the same star.

(a) Instrumental and photographic errors. Slight errors in focussing the spectrograph, provided they do not cause blending of close lines, should not seriously affect measures of total intensity. This also applies to errors due to limited resolving power and temperature shifts which render the lines shallower and more diffuse, but leave the total amount of light absorbed unchanged. The slit-width normally used was $m=2$, *i.e.* with the second diffraction band just outside the field; exposures of 90 minutes or more were liable to suffer from temperature shifts.

The sources of photographic errors are very numerous. Errors in the wedge constant will give systematic errors affecting directly all measures. The wedge was calibrated before work was begun by three independent methods. The first method (that described by H. H. Plaskett *) employed a series of apertures which reduced the light source in front of a spectrograph in known ratios: the second a "raster," † for which monochromatic light had to be used: the third was similar to the first, but a different spectrograph was used and a variable slit-width instead of varying apertures. The results by the three methods are shown in Table I.

TABLE I

Wedge Constant

(Wave-lengths 4600-6400)

	λ (100 Å.)	46	48	50	52	54	54.6	56	57.8	58	60	62	64
Method I (15 plates)	0.220	.211	.197	.194	.193	...	.190	...	.195	.202	...	...	...
		± 1	± 1	± 1	± 1	± 1		± 1		± 1	± 1		
„ II (5 plates)	...	...	...	...	...	...	.198	...	.198	...	...	...	...
							± 4		± 4				
„ III (4 plates)	0.229	.219	.186	.182	.183	...	.197	...	.196	.210	.212	.212	
		± 12	± 4	± 4	± 8	± 9		± 9		± 5	± 5	± 8	± 8

The m.s.e. of all the separate probable errors is 0.0055, so that the probable error of the weighted mean may be taken as 3 per cent. It is to be noticed that the p.e. in the first method is less than 1 per cent. owing to the large number of plates. This illustrates the danger in photometric work of putting too much trust in the results obtained with one instrument. Fortunately, in the work with which we are concerned at present systematic errors of this nature will not seriously trouble us, for we are measuring *relative* rather than *absolute* intensities.

The most dangerous form of photographic error is Eberhard effect, but brush-development seems sufficient at any rate to diminish the effect so that it is no larger than other accidental errors. The Hypersensitive

* M.N., 91, 877, 1931.

† Woolley, *Annals S.P.O. Cambridge*, 3, 2.

plates, however, seemed to show a definite tendency to suffer from this defect. Some photometric errors might arise from the fact that the standard spectrum was continuous and the stellar spectrum intermittent, owing to the star being trailed along the slit; moreover, owing to the same cause, the length of exposure time at any one point of the spectrum had to be guessed as a fraction of the total exposure time. However, these errors are likely to be small; the intermittency effect was tested and found to shift the characteristic curve bodily without changing its shape, so that this will not upset measures of relative intensity; the same applies to exposure time, for it has been claimed that altering the exposure by a factor of 1000 leaves the contrast γ unchanged.

Another possible source of error is the latent image effect noted by Keenan * and others, in which the density and characteristic curve undergo small changes between exposure and development. This effect has been tested and found to be by no means negligible. For a Process Panchromatic plate developed for four minutes in M.Q. at 60° F. γ appeared to change by 5 per cent. and 6½ per cent. in three and two hours respectively between exposure and development. Results are shown in Table II. Four plates were used to each test and measures made at three wave-lengths, so that each figure is a mean of twelve and the effect should be well clear of accidental errors.

TABLE II

Latent Image Effect

(Ilford Rapid Process Panchromatic Plates)

Change of density D at different densities

(The figures in the second row give times of exposure before development)

Micro- photometer Reading	log D (1st Series)		$\Delta \log D$	log D (2nd Series)		$\Delta \log D$
	2.2 h.	0.1 h.		3.2 h.	0.1 h.	
30	0.394	0.393	0.001	0.399	0.395	0.004
40	.336	.324	.012	.338	.330	.008
50	.278	.262	.016	.278	.265	.013
60	.213	.197	.016	.221	.206	.015
70	.148	.116	.032	.159	.134	.025
80	.054	.024	.030	.076	.043	.033

The effect is large enough to cause considerable trouble in photometric work, and must clearly be avoided as far as possible. It cannot affect the measures in the present paper seriously as little time was allowed to lapse

* *Ap. J.*, 74, 105, 1931.

between the stellar and wedge exposures, and development was often delayed in order to decrease the differential effect.

(b) Microphotometer errors are not likely to be very serious. The slit-width used (0.014 mm.) gives a resolving power larger than that of the original spectrograph. In taking tracings the plate is levelled so that the plate-grain is in focus everywhere, but the writer has found that readings are very insensitive to focus, a large focussing error only affecting the reading by about 2 per cent. Lack of sensitivity has caused some trouble, but usually it is negligible, for tracings taken at different speeds of the motor have given identical results. A creep in the zero has also been occasionally noticed, but this usually disappears after the lamps have been left on half an hour or so.

(c) The largest errors must arise in the final reduction of the tracings. The drawing-in of the continuous background is a notoriously difficult task in any spectrophotometric work, and in many late-type stars with plentiful lines and occasional bands it becomes almost hopeless. However, in this respect the green region is better off than the violet, where lines are more numerous.

Many of the lines measured were very narrow, and at the dispersion used (50–60 Å./mm.) few were even wider than 5 mm., so that the actual width measurement was subject to a large accidental error, which could be greatly aggravated by coarse plate-grain. Finally there was the effect of blends which in many cases must be responsible for larger errors than any hitherto mentioned. Efforts to allow for blends will be described later.

Table III shows a rough quantitative estimate of the part played by each of the above-mentioned sources of error in the measurements.

TABLE III

	Per cent.
Spectrographic errors (focussing, etc.)	3
Characteristic curve (including wedge constant)	4
Photographic errors : Eberhard effect	5
Latent image	4
Exposure time	
Intermittency, etc.	
Microphotometer errors : Sensitivity, unsteadiness, etc.	3
Reduction : Determination of "clear glass" reading	3
" " continuous background	6
Width measurement (including grain effects)	10
Blends	15

Taking these as independent sources of error the total probable error of a single measurement is 21 per cent. (30 per cent. corresponds to about one Rowland number for an average line). The last three errors depend very largely on the strength of the line and the presence of neighbouring lines ; blends may in some cases provide more than 50 per cent. of the measured width, and for lines less than 0.1 Å. wide the accidental error of width measurement swamps all others as the line is practically lost in the grain ;

such lines were only measured when of special interest (*e.g.* the weakest members of the *Fe* multiplet and the *D* lines in early-type stars).

Table IV shows results of measures of total intensities (in angstroms) of solar lines by some observers. The writer wishes to thank Mr. Evershed for the privilege of the use of his solar equipment at Ewhurst. The other two unreferenced columns are also by the writer with Cambridge instruments.

TABLE IV

<i>Mg</i>	Unsöld *	Korff †	Plaskett ‡		Min-naert §	Ewhurst	McClean, Littrow	Low Dispersion	
			Centre	Limb				Huggins (Sky)	Shajn (Moon)
5183	...	...	1.54	1.14	1.50	...	...	1.23	1.65
5172	...	...	1.08	0.83	1.14	...	...	1.04	...
<i>Na</i>									
5890	0.66	0.58	...	...	...	0.63	0.65	0.87	1.04
5896	0.46	0.44	...	...	...	0.57	0.53	0.63	0.84
<i>Fe</i>									
5371	...	...	...	...	0.29	0.28	...	0.28	...
5397	...	...	...	...	0.22	0.19	...	0.23	...
5434	...	...	...	...	0.23	0.19	...	0.17	...
5446	...	...	...	...	0.31	0.22	...	0.22	...
5497	...	...	...	...	0.16	0.12	...	0.13	...
5501	...	...	...	...	0.14	0.11	...	0.08	...
5506	...	...	...	...	0.15	0.11	...	0.11	...

The last two columns are measures with stellar spectrographs, *i.e.* much smaller dispersion than for the others made with solar apparatus. The errors are on the whole no larger than we would expect, though the *D* lines seem to appear consistently too large with low dispersion. This may be due to Eberhard effect, especially as the lines were not thoroughly resolved. It is to be emphasised that a systematic error affecting all measures in the same proportion will not affect measures of relative intensity; what we have to be on our guard against chiefly is an error which will affect strong and weak lines in different proportions. At present, when even the high-dispersion measures may be in error by 5 per cent. or more for the weak lines, it is difficult to judge whether there are any such errors affecting our present results.

Table V shows the results for the *D* lines for α Ori. and α Tau. and illustrates the order of agreement to be expected in results from different plates for typical strong lines.

§ 3. *The Interpretation of Equivalent Widths.*—In comparing observation with theory we have followed the usual practice of interpreting equivalent

* *Zeits. für Phys.*, 46, 765, 1928.

† *Ap. J.*, 76, 294, 1932.

‡ *Op. cit.* § *Zeits. für Ast.*, 1, 197, 1930.

|| *M.N.*, 91, 694, 1931.

The first three observers publish contours only; the figures in the table are planimeter measures by the writer from these contours. The figures given for Unsöld and Korff being based on diagrams may be slightly less accurate than if figures were available.

TABLE V

<i>a</i> Ori.		<i>a</i> Tau.	
<i>D</i> ₁	<i>D</i> ₂	<i>D</i> ₁	<i>D</i> ₂
82	99	98	132
66	78	91	140
84	84	99	142
72	88	106	145
66	77		
74.0 ± 5.8	85.2 ± 6.0	98.5 ± 4.1	139.8 ± 3.6

widths W in terms of the number N of atoms active in producing the line, by means of the relation

$$W \propto \sqrt{N}. \quad (3.1)$$

It will not perhaps be out of place to give this law some justification.

The absorption coefficient has been given by Weisskopf and Wigner * as

$$\kappa_\nu = \frac{f_{ki} \frac{ne^2}{4\pi m \nu} \delta_{ik}}{(\nu - \nu_{ik})^2 + \delta_{ik}^2}, \quad (3.2)$$

where n is the number of atoms per c.c., f_{ki} is the oscillator strength, i, k being the lower and upper states respectively concerned in the production of a line of frequency ν , and the half-width δ_{ik} is given by

$$\delta_{ik} = \frac{1}{4\pi} \left[\sum_n A_{in} + \sum_m A_{km} \right], \quad (3.3)$$

the summations being taken over all states n below i in the first case, and all states m above k in the second. We may usually neglect δ_{ik} in the denominator of (3.2) except at the centre of the line (e.g. for *Na* 5890, $\delta = 3 \cdot 10^7$, $\nu = 5 \cdot 10^{14}$, and for wave-lengths more than 0.005 Å. from the line-centre

$$\frac{\delta^2}{(\nu - \nu_0)^2} < 0.01).$$

In Sir Arthur Eddington's † theory of the formation of absorption lines the ratio r_ν of light emitted of frequency ν to the continuous emission outside is given as a function of the parameters α, β appearing in the equation

$$\frac{1}{1 + \eta} = \alpha + \beta\tau, \quad (3.4)$$

the assumed law of variation with optical depth τ of η , defined as the ratio of mass coefficients of line absorption l_ν and continuous absorption k_ν .

* *Zeits. für Phys.*, **63**, 54, 1930; also Weisskopf, *Observatory*, **56**, 302, 1933.

† *M.N.*, **89**, 620, 1929.

The coefficient l_ν is simply $4\pi\nu\kappa_\nu$, so that we have from (3.2) except at the centre of the line

$$\begin{aligned}\eta &= \frac{e^2}{m} \frac{nf_{ki}\delta_{ik}}{k_\nu(\nu - \nu_{ik})^2} \\ &= C \frac{nf_{ki}\lambda^4\delta_{ik}}{k_\nu(\lambda - \lambda_{ik})^2},\end{aligned}\quad (3.5)$$

C being a constant.

Now the parameters α, β in (3.4) depend on the value η_0 taken by η at a particular depth τ_0 , so that r_ν is a function of η_0 . If r_ν is the same function of η_0 for two lines, the widths $(\lambda - \lambda_{ik})$ at a given r will be proportional to $\sqrt{\frac{nf_{ki}\lambda^4\delta_{ik}}{k_\nu}}$. The variation of η with τ can be calculated theoretically as Woolley has done; it is the same for all lines due to the same element in the same state of ionisation. Woolley found further* that with $\eta = 1$ at $\tau = \frac{1}{3}$ the corresponding r is the same for different elements (0.71 for neutral metals, 0.66 for ionised metals).

The continuous absorption coefficient k_ν is proportional to $\frac{\lambda^3}{T^{\frac{3}{2}}}$ † so that $\frac{\lambda^4}{k_\nu}$ will not vary appreciably when comparing lines within the same region of the spectrum. A further simplification for multiplets is that within a multiplet δ_{ik} is constant, so that in this case, for given r_ν

$$(\lambda - \lambda_{ik}) \propto \sqrt{nf_{ki}},$$

nf_{ki} being directly proportional to the number of atoms active in producing the line, it follows that

$$W = 2 \int_0^1 (\lambda - \lambda_{ik}) dr \propto \sqrt{N},$$

which shows that the square root law (3.1) should hold at any rate within multiplets.‡ This assumes that the line contour is entirely due to the absorption coefficient (3.2). The Doppler effect will modify this coefficient especially for weak lines, although according to Weisskopf § the modification is negligible in the thick layers of a stellar atmosphere. The finite resolving power of the spectrograph modifies the contour, but should leave the total absorption unchanged. The above argument shows that outside a multiplet

$$W \propto \sqrt{N\delta_{ik}}$$

should be used, and this may fail for comparison of ionised and neutral lines.

* *M.N.*, 92, 497, 1932.

† Pannekoek, *M.N.*, 91, 163, 165, 1930.

‡ Within the multiplet the quantity N is simply proportional to $q_i f_{ki}$, and the law which is actually tested is $W^2 \propto q_i f_{ki}$, which is true whatever the distribution. q_i is the statistical weight of the lower level.

§ *Observatory*, 56, 307, 1933.

§ 4. *Results*.—The study of multiplet intensities and comparison with theory is of interest in view of the fact that Russell and Adams's calibration of Rowland's scale was based on the theoretical intensities, and further that departures from the theoretical intensities have recently been observed by Minnaert* and Woolley.† The *Fe* multiplet discussed in this paper was well-suited to a study of this nature in view of its strength, large range of intensity (100 : 1), and also to the advantage of lying in the green region of the spectrum where the relative scarcity of lines both decreases the effect of blends and lessens the difficulty of drawing in the continuous spectrum.

In applying (3.1) we have to allow for the distribution of atoms in the different multiplet states which vary slightly in excitation potential. If N_s is the theoretical intensity of a line arising from a state s with excitation potential χ_s , we should have agreement with theory if

$$\log N_s = 2 \log W'_s = 2 \log W_s + 2\Delta \log W_s + .4343\chi_s/kT, \quad (4.1)$$

$\Delta \log W_s$ being the correction for Doppler broadening. In practice we plot the observed values $2 \log W'_s$ against $\log I_s$, draw a straight line through the points and measure the slope r given by the mean value of $\frac{2 \log W'_s}{\log I_s}$.

Table VI (a) shows the measures of W for each line of the multiplet for individual stars (in units of 0.01 A.). The accidental errors were considered too large to derive reliable values of r for each star, so they were grouped into spectral types. Doppler corrections $2\Delta \log W$ were taken from Woolley's calculations at $\lambda 5200$, but these only affected the weakest lines, and in the latest types were negligible even for these lines.

Table VI (b) shows the mean values of $2 \log W'$ derived from (4.1) for mean types F5, G0, K0, K5 and M. Fig. 1 shows these values plotted against $\log N$. The slopes of the lines ‡ drawn through the points were determined by the method of least squares, the weakest three lines (subject to the largest accidental errors) being grouped together as a single mean point for this process. It is seen at once that there is fair agreement with theory for types F5, G0, K0, but in the coolest stars the weak lines are relatively strengthened. This phenomenon appears to be particularly marked in α Orionis, which gives an individual value of $r = 0.60$.

It remains to consider whether this can be a similar effect to that already observed in some solar multiplets. As mentioned above the Doppler corrections have only had to be applied to the earlier type stars, and in any case these corrections are probably on the large side, as according to Weisskopf Doppler broadening is negligible in stellar absorption lines, so that this cannot be the explanation of the anomalous behaviour in the late types. It is worth noting that in late types the danger of drawing in the continuous too low is increased, but if it is systematically drawn in too low for the whole multiplet the weak lines will suffer most, so that this error works in the opposite direction to the observed tendency. However, there is a danger

* *Zeits. für Ast.*, 1, 192, 1930.

† *Ap. J.*, 72, 256, 1930.

‡ They are the lines of regression of $2 \log W'$ on $\log N$.

TABLE VI

Fe Multiplet (5269-5506)(a) *Total Intensities (unit 0.01 A.)*

	5269	5328	5371	5397	5405	5429	5434	5446	5455	5497	5501	5506	No. of Plates	
F	η Boo	82	49	47	29	59	49	33	53	26	...	...	13	1
	α Per	74	55	31	30	38	23	17	22	17	13	5	5	3
	α Can Min	37	32	10	...	...	...	...	...	...	...	...	...	2
	α Urs Min	50	57	34	19	25	15	24	27	24	...	...	...	2
	γ Cyg	117	86	48	46	52	44	29	37	29	11	4	11	2
G	Sun	83	60	28	23	31	28	17	22	21	13	7.5	11	3
	α Aur	89	82	48	38	50	29	26	37	25	10	10	15	3
	ϵ Leo	80	46	21	21	18	17	12	21	12	...	...	...	1
	ϵ Gem	101	63	61	51	82	43	52	41	17	9	...	...	2
	η Dra	92	98	40	24	39	50	14	36	16	7.5	6	7.5	1
	β Dra	76	51	65	57	47	29	...	26	25	...	...	...	1
	μ Her	82	66	22	19	67	...	...	...	...	...	...	...	2
K	β Gem	127	108	61	44	66	46	35	39	32	7.5	7.5	14	5
	α Boo	122	102	55	45	53	34	29	43	32	9	6	15	4
	α Urs Ma	111	99	61	48	62	43	24	24	28	7.5	7.5	12	3
	γ Leo	122	96	43	54	65	19	18	34	28	7.5	9	14	3
	α Hya	105	138	56	39	51	40	24	55	46	6	7.5	28	1
	γ And	147	96	84	86	111	64	49	58	49	23	5	16	1
	ϵ Aur	185	172	55	84	111	49	55	58	58	...	...	...	1
	ϵ Cyg	93	85	42	50	39	23	19	30	15	...	...	...	1
	α Ari	118	89	71	43	64	38	36	29	33	...	...	...	2
	α Tau	188	161	99	85	103	58	47	57	56	16	14	29	4
	β Urs Min	143	158	97	77	99	50	58	65	54	18	24	28	2
γ Dra	156	143	77	72	88	36	37	46	49	23	15	34	2	
M	α Ori	168	159	119	110	113	72	83	108	51	35	40	67	4
	β And	167	146	97	87	90	53	53	49	52	19	14	20	2
	α Cet	179	144	46	35	69	31	29	...	40	14	14	7.5	1
	μ Gem	200	130	65	97	65	31	48	...	...	...	...	...	1

(b) *Mean 2 log W' (corrected for Doppler Effect)*

Mean Type													
F5	1.52	1.40	0.96	0.79	1.18	0.69	0.66	0.89	0.61	1.34	2.70	2.95	
G0	1.91	1.77	1.29	1.06	1.36	1.08	0.74	1.09	0.83	1.80	1.00	1.91	
K0	2.13	2.09	1.62	1.41	1.73	1.22	1.10	1.30	1.18	1.14	2.91	0.34	
K5	2.39	2.38	2.09	1.80	2.22	1.57	1.67	1.67	1.70	0.73	0.44	1.26	
M	2.48	2.46	2.16	2.01	2.24	1.69	1.82	1.92	1.76	1.17	1.02	1.38	
log N (theoretical intensity)	2.34	2.18	1.98	1.48	1.75	1.62	1.45	1.60	1.45	0.60	0.30	0.60	

in the case of the present multiplet of the two strongest lines being measured too weak owing to the presence of a *TiO* band at 5167. But this should not affect the slope r as determined from the other lines, and this is still less than unity.

As remarked in our discussion of errors, blends form the greatest danger in measurements of stellar intensities, and, further, might easily cause large changes of relative intensities of lines in different stellar types. An

attempt was therefore made to correct for blends as they appear in different types with the help of Dunham's spectrum of α Persei * (F5), and Rowland's tables for Sun (Go) and Spot spectra. It was assumed that a measured line would include 90 per cent., 50 per cent. and 20 per cent. of blends at distances 1, 2 and 3 angstroms respectively; this empirical scale was found to reduce the solar measurements to values agreeing satisfactorily with high-dispersion measures. Rowland intensities given in the tables were transferred to

FIG. 1.—Behaviour of Multiplet with Spectral Type.

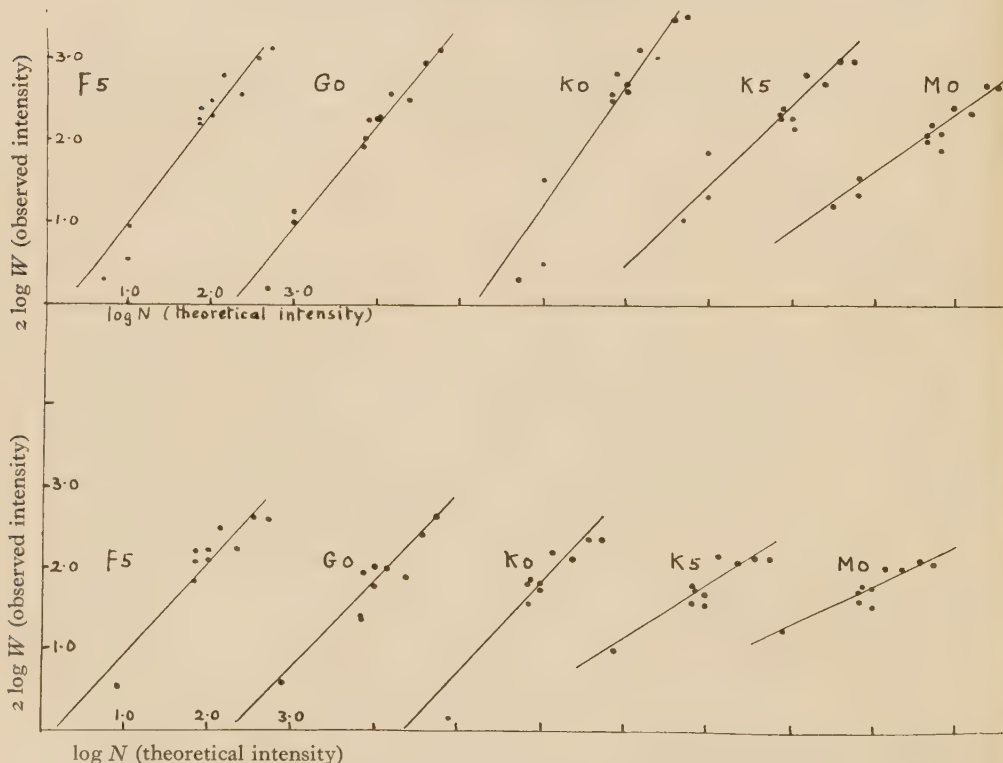


FIG. 2.—Fig. 1 with Estimated Blend Corrections.

equivalent breadths by the use of Woolley's calibration of Rowland's scale (*Ann. S.P.O., Cambridge*, vol. iii, pt. 2). The spot spectrum was assumed to correspond to stellar type Ko, and hence we could estimate blend corrections for types F5, Go, Ko separately; these are shown in the accompanying table. The correction in the last column was also applied to types K5 and Mo.

It will be seen that in general the strongest lines suffer most from blends, the weakest hardly at all, so that the result of applying the corrections is to decrease r further, as can be seen at once in fig. 2, which is similar to fig. 1 but using $2 \log W'$ corrected for blends. Table VIII illustrates these results numerically, giving the values of r with and without blend corrections,

* Contributions from the Princeton University Observatory, No. 9.

TABLE VII

Estimated Blend-corrections ($-2\Delta \log W$) for Different Types

	$\log N$	F5	Go	Ko (Spot)
5269	2.34	0.52	0.30	0.50
5328	2.18	.38	.38	.46
71	1.98	.32	.44	.24
97	1.48	.15	.16	.30
5405	1.75	.31	.39	.27
29	1.62	.10	.10	.25
34	1.45	.22	.38	.30
46	1.60	.38	.38	.23
55	1.45	.38	.48	.13
97	0.50	.10	.00	.05
5501				
06				

and also the values of R the correlation coefficient between $2 \log W'$ and $\log N$.

TABLE VIII

	Without Blend Correction		With Blend Correction	
	r	R	r	R
F5	1.31	0.96	1.11	0.93
Go	1.25	.92	1.08	.98
Ko	1.42	.94	1.20	.95
K5	0.88	.94	0.65	.90
M	0.72	.93	0.49	.88

The change in R might be used as an indication of the accuracy of our blend corrections. The scatter is clearly reduced in Go, but in most of the other types there is a slight decrease in R , so that blends may still be a factor in the change of r with the spectral type. It is unfortunate that we cannot make a better estimate of the blends in types K5 and M, but it seems very unlikely that the weak lines are here affected by blends more than the strong lines.

There appears to be no systematic change of r with absolute magnitude judging from our observations of G5 to K5 stars when grouped by absolute magnitude.

§ 5. *Solar Multiplets*.—It was originally intended to study stellar spectra of the same stars in the violet region taken with the 2-prism spectrograph of the Newall telescope, and find whether other multiplets showed similar behaviour in late-type stars. Unfortunately, however, the lines were found to be so numerous that, in spite of the greater dispersion, the errors of blends and of the drawing-in of the continuous were too large to derive reliable results; and, further, the strongest lines belonged to multiplets whose intensities were unknown theoretically.

It was therefore decided to fall back upon the data available from the Rowland numbers of disc and spot spectrum in the Sun, to see whether multiplets behaved in any way similar to the change observed from stellar type K to M. In the course of the work it was noted that multiplets of low-excitation potential consistently showed the most anomalous behaviour with r small, in fact r seemed to increase systematically with excitation potential. The multiplets were therefore grouped by excitation potential and mean values of r derived for each group. The method was simply to tabulate $\log I$ (the theoretical intensity) and $B \log A$ (from the Russell-Adams calibration), and calculate $\Delta(\log I)$, $\Delta(B \log A)$, the deviations from the mean of each multiplet. The average value of $\Delta(B \log A)$ for a given $\Delta(\log I)$ was then taken for each group of multiplets, and these means plotted against $\Delta(\log I)$ gave values of r . Rowland numbers were taken for both Sun and spot from Miss Moore's "Multiplet Table of Astrophysical Interest."

Results are given in Table IX for different elements. The heights given

TABLE IX

Change of $r \left(= \frac{\log(\text{observed intensity})}{\log(\text{theoretical intensity})} \right)$ in Multiplets from Disc to Spot

(Based on Rowland numbers and Russell-Adams calibration)

Element	Number of Multiplets	Mean E.P. (Lower State)	r_d Disc	r_s Spot	r_s/r_d	Mean ($B \log A$) Disc	Spot	Height (km.)
<i>Fe</i>	3	0.0, 0.9	0.25	0.00	0.00	2.60	3.16	600-1000
	3	1.5	0.58	0.45	0.78	3.02	3.27	800
	6	2.3	0.82	0.60	0.73	2.10	2.37	500
	5	2.6	0.82 :	0.82 :	1.00 :	1.71	1.88	450
	5	2.8	0.87	0.75	0.86	1.53	1.84	430
	4	3.3	1.05	1.05	1.00	1.62	1.98	400
	3	3.8	1.46	1.46	1.00	1.08	1.12	360
	3	4.2	1.06	1.23	1.10 :	1.02	0.92	340
<i>Fe</i> ⁺	5	2.8	0.66 :	1.00	1.52 :	1.45	0.66	1000
<i>Ca</i>	3	1.9	0.72	0.58	0.81	2.32	2.96	700
	7	2.5	1.10	0.63	0.57	1.35	2.28	450
<i>Ti</i>	5	0.0	1.10	0.50	0.45	1.07	2.36	400
	2	0.8	0.90	0.85 :	0.95 :	1.17	2.27	400
	8	1.2	1.21	1.12	0.93	0.27	1.25	400
	3	1.7	1.18	1.18	1.00	0.20	1.18	350
<i>Ti</i> ⁺	5	2.0	1.10	0.90	0.82	0.25	0.90	300
	6	0.0	0.75 :	...	...	2.52	...	1000
	8	1.3	0.86	0.82	0.95	1.67	1.46	1000
<i>Cr</i>	4	1.0	1.14	0.92	0.81	1.65	2.48	500
	5	2.8	1.14	1.13 :	0.99 :	0.51	1.07	400
<i>V</i>	4	0.2 (-2.5)	0.70 :	0.74 :	1.06 :	-0.40	1.77	450
	4	0.2 (-3.1)	1.00	1.10	1.10	0.55	1.75	450
<i>Mn</i>	4	2.7	1.25	1.30	0.96	1.31	1.80	400

in the last column are necessarily approximate, being based on Mitchell's* figures for particular lines, and typical strengths and excitation potentials. In the majority of cases there is a definite decrease of $\bar{r}$ from disc to spot, especially in the low-potential multiplets of iron (the $a^5F - z^5D^0$ multiplet observed in stellar spectra is one of the first group with E.P. 0.9-3.2). It has been noted by several observers† that the strengthening in sunspots of weak lines is greater than that of strong lines; the above table shows that this holds within multiplets. Another striking feature of the table is the steady increase of r with E.P. for the sequence of *Fe* multiplets. Comparison with other elements suggest that small values of r are associated with great height in the chromosphere rather than directly with excitation potential. Anomalies do not, however, appear to be a necessary consequence of great height for the simpler multiplets (e.g. $Mg3^3P^0 - 3^3D$ rising to about 2000 km. appears to be fairly normal).

§ 6. *Discussion.*—The observational data indicate that the weak members of a multiplet strengthen more rapidly than strong lines (i) from stellar types K to M; (ii) from disc to spot spectrum for low-potential multiplets; and (iii) the anomalies increase with height in the atmosphere.

Collisions have been suggested as a possible explanation of anomalous multiplet intensities, but this is not supported by the above data. The pressures in the atmospheres of M giants must be considerably smaller than in K stars, and the low-potential multiplets arise, as Mitchell has shown, at an average height higher above the photosphere, and therefore at a lower pressure. In both cases we expect collision effects small where the anomalies are observed.

Another possible explanation is that the anomalies arise through misinterpretation of the equivalent widths in terms of number N of atoms at work (e.g. by neglecting Doppler broadening). There are two methods at present for deriving N , viz.:

(a) Measurement of equivalent breadths, and use of the square-root law—this as shown above depends on the theoretical absorption coefficient and the assumption that disturbing factors such as the Doppler effect can be neglected or have been allowed for.

(b) Use of the Rowland scale and the Russell-Adams calibration.‡ This depends on the multiplet intensities following the theoretical intensities in the mean.

Minnaert§ has published a diagram based on his measurements of solar lines in different spectral regions, in which he effectively plots the number of atoms derived from the two methods (a) and (b) against each other. The fact that he did not obtain a straight line inclined at 45° to the axes indicates a failure either of the square-root law or of the Russell-Adams calibration, or both. Minnaert's curve was steeper than 45° for strong and weak lines, whilst for intermediate lines (Rowland numbers 1 to 8) there was

* *Ap. J.*, **72**, 160, 161, 167, 1930.

† Hale and Adams, *Ap. J.*, **23**, 29, 1906; C. E. Moore, *Ap. J.*, **75**, 315, 331, 1932.

‡ *Ap. J.*, **68**, 1, 1928.

§ *Zeits. für Ast.*, **1**, 192, 1930; **2**, 165, 1931.

a point of inflexion with a slope of 0.67 (similar observations by Woolley give a slope of 0.61 for intermediate lines). He explains this as a failure of the square-root law for lines weaker than Rowland number 8, just as Schutz* had found a failure of the law in the laboratory below a certain pressure. This explanation is not upset by the present results so far as anomalies in solar (disc) multiplets are concerned, but it cannot be extended to account for the observed change in r from K to M, and from disc to spot. The lines of the *Fe* multiplet ($a^5F - z^5D^0$) are in both cases strong—average Rowland number 8 or more—and increasing in strength, so that from Minnaert's curve we should expect r to increase, but a decrease is actually observed.

A third possible explanation of anomalous multiplet intensities is that it is a phenomenon of self-reversal which weakens strong lines and strengthens weak ones (this would happen in a state of non-thermodynamical equilibrium if the principle of detailed balancing is upset in such a way that there is a cyclic motion of electrons through the different multiplet states). This explanation seems to fit in best with the observations, since departure from thermodynamical equilibrium is most likely in sunspots and in the upper layers of the solar atmosphere. Further, it is well known that cool giants often show bright lines, indicating either an extended chromosphere or a non-steady state with lack of thermodynamical equilibrium.

The type of phenomenon considered here underlying the multiplet anomalies is that of interlocking as described by Woolley; † the number of upward transitions between any two states is dependent on the intensity of the incident radiation of the corresponding wave-length. High up in the atmosphere where the radiation is deficient in certain line-frequencies (the deficiency being greatest for strong lines) the number of upward transitions is relatively diminished for strong lines; but the number of downward transitions is determined by constant transition probabilities (Einstein "stimulated emission" being negligible except in the hottest stars), and the ratio of such transitions from a given state for strong and weak lines within a multiplet remains constant. Thus the strong lines have a greater tendency towards self-reversal than weak lines—in our notation $r < 1$.

§ 7. *Note on the Russell-Adams Calibration and Ionisation Temperatures.*—The discordance found by Minnaert between the two methods of deriving $\log N$ might be attributed to a failure of the Russell-Adams calibration instead of to a failure of the square-root law. The scale of the calibration depends directly on the mean value of r for multiplets used in the calibration, which was based on the assumption that $\bar{r} = 1$. Since the multiplets are limited in range of intensity the scale at a particular Rowland number n is dependent on $r(n)$, the mean value of r for multiplets of average strength n . Then the usual expression for N has to be modified to

$$\log N = r(n) \cdot B \log A. \quad (7.1)$$

As noted above, on theoretical considerations the effect of interlocking is to make values of $r < 1$ not only possible but probable, whilst values of $r > 1$

* *Zeits. für Ast.*, I, 300, 1930.

† *Ap. J.*, 73, 200, 1931.

are more difficult to explain. Further, observations of equivalent widths show a majority of multiplets with $r < 1$. It therefore seems not unlikely that we should apply a factor $r(n) < 1$ as in (7.1) to the Russell-Adams calibration. The value

$$r(n) = 0.65, \quad (7.2)$$

for Rowland number $n = 1$ to 8, is sufficient to remove the discordance between this method of deriving $\log N$, and that from the measurement of equivalent widths with the use of the square-root law. To obtain a fit for stronger and weaker lines a value of $r(n) > 1$ has to be assumed: but at these extremes there may well be another factor at work.

Assuming the existence of a factor $r(n)$ as in (7.1) different from unity, the values of r given in Table IX based on the Russell-Adams calibration are erroneous, and have to be multiplied by $r(n)$ to give the true values of r . But the steady increase of r with decreasing strength shown by the *Fe* multiplets in Table IX may be used to show that $r(n)$ probably changes with n . Assume the Russell-Adams calibration correct and that $r(n) = 1$ for all n ; then the change of r with strength for the *Fe* multiplets has to be compensated for by a change of r in the opposite direction for other elements. It would be rather a happy chance if r changed with strength in different directions for different elements in such a way that the mean r was always unity; by assuming a correcting factor $r(n)$ which increases with decreasing n , the change of r with strength would be in the same direction for all elements.

Such a variation of $r(n)$ with n may have an important bearing on the ionisation temperatures derived by Russell and Adams using their calibration of Rowland's scale. It will be recalled that the method depended on the determination of the slope of the function Y plotted against excitation potential E_x , where

$$Y = \log \frac{N'}{N} = \frac{B \log A'}{B \log A}. \quad (7.3)$$

The correction (7.1) modifies this to

$$Y_c = \frac{r(n')}{r(n)} \cdot Y. \quad (7.4)$$

The function r depends only on anomalies in solar multiplets, but the strength n of a solar line will change considerably when compared with a low-temperature star like α Orionis, especially for small E_x . It is therefore for these lines that the correcting factor $\frac{r(n')}{r(n)}$ may become important. We have been led to expect that it will decrease from unity with decreasing E_x . The result of applying a correction as in (7.4) with $\frac{r(n')}{r(n)} < 1$, to lines with small E_x would be to straighten out the well-known curve* obtained by Adams and Russell, which they attributed to lack of thermodynamical equilibrium at high potentials. Their empirical correction effectively

* *Ap. J.*, 68, 20, 1928.

measures the slope of the curve at $E_x = 0$; the present considerations suggest that this slope may be too steep and that the true slope should be measured at some undefined intermediate potential. The result would be to raise the temperatures of the coolest stars. The same trouble may be present in Miss Moore's similar curve * for obtaining the sunspot temperature; she found that the empirical correction for *Fe* lines alone gave a temperature (4290) considerably lower than her final determination of 4720 based on general considerations of all elements. The measurement of the slope at an intermediate potential (2.7 volts) removes this discrepancy.

When sufficient measures of equivalent widths become available the correction (7.4) will no longer apply, but multiplet anomalies may still remain a disturbing factor in the measurement of ionisation temperatures. For lines of the same multiplet—and therefore of the same E_x —which strengthen at different rates between any two stars, the values of the ratios Y may be considerably different. If the lines were grouped by strength, each group would give a different temperature and the adopted temperature would depend on the method of averaging the Y 's. If lack of thermodynamical equilibrium is causing disturbances at both high and low potentials, the definition of temperature begins to be a difficult task.

Summary.—Results of a spectrophotometric study of 30 late-type stars are collected and discussed. The various sources of error are considered in detail and an estimate of 20 per cent. given as the probable error of an average line; blends and other factors may increase errors further in some cases, especially in faint lines. A less-known photometric error due to growth of the latent image has been found to produce an error of 5 per cent. in certain conditions.

The $a^5F - z^5D^0$ multiplet of *Fe* consisting of twelve members (5269–5506) is measured, and there appears to be a marked strengthening of the weak lines (Minnaert-Woolley effect) in K5 and M stars. Corrections for blends estimated from the solar spectrum only increase this tendency.

The same phenomenon has been found in comparing the sunspot and solar disc spectra, and it appears that the anomalies are most frequent for strong lines of low excitation potential formed high in the atmosphere. Of the possible explanations, interlocking, involving a lack of thermodynamical equilibrium, seems to fit the facts best.

The interpretation of equivalent widths in terms of the number of active atoms is discussed and compared with the alternative method of using the Russell-Adams calibration of Rowland's scale. It is shown how a correction may have to be applied to the latter method as a result of anomalous multiplets which would reconcile the present disagreement between the two methods.

Such a correction might have an important bearing on ionisation temperatures obtained by Adams and Russell's method—chiefly in the direction of raising the temperatures of late-type stars.

* *Ap. J.*, 75, 325, 1932.

It is a pleasure to record my warmest thanks to Professor Stratton for allowing me every facility for the work and for his constant encouragement ; to Dr. Woolley for suggesting this branch of stellar work ; also to Sir Arthur Eddington, Professor H. H. Plaskett and Dr. R. O. Redman for some highly valued advice and criticisms.

*Solar Physics Observatory,
Cambridge :
1933 November 29.*

THE ATMOSPHERICAL EXTINCTION FACTOR IN PHOTO-ELECTRIC PHOTOMETRY.

W. M. Smart, M.A., D.Sc.

(Communicated by the Director, Cambridge Observatory)

1. In precise stellar photometry a necessary observational quantity required in the subsequent reductions is the atmospherical extinction factor. If this factor is known, the observed difference of magnitude between a variable star and a suitably chosen comparison star—the latter being required as the photometric standard of reference—can be converted into the true difference of magnitude which would be measured if our observations could be conducted outside the terrestrial atmosphere. The extinction factor varies, of course, from night to night and, in an uncertain climate, generally from hour to hour. It is also dependent on the spectral types of the stars observed ; hence it is essential to choose a comparison star of the same spectral type as the variable star. A second desirable condition, if the reliability of the final results is to be commensurate with the extreme accuracy of the present-day photo-electric photometer, is that the variable and the comparison star should be observed at nearly the same zenith distance. If observations of a particular variable are to extend over several hours this second condition requires the choice of a comparison star within a few degrees in the sky of the variable. Under these circumstances an approximate value of the extinction factor is generally good enough. This value may be estimated from the state of seeing or from the performance of the photometer or derived from observations specially undertaken for this purpose.

2. The effect of atmospherical extinction on the magnitude of a star is given, for moderate zenith distances, by the formula

$$m - m_0 = f \sec z, \quad (1)$$

where m is the magnitude of the star as viewed through the atmosphere, m_0 is the magnitude it would have if observed outside the atmosphere, z is the star's zenith distance and f the extinction factor. I shall call m_0 the *true* magnitude. In section 3, a more general formula, applicable to zenith distances of 70° or 75° , will be investigated. Suppose (1) to refer to the

comparison star. If M , M_0 and Z are the corresponding quantities for the variable star, we have

$$M - M_0 = f \sec Z. \quad (2)$$

Denote by Δm the difference, $M_0 - m_0$, of the true magnitudes of the two stars. Then, by (1) and (2),

$$\Delta m = M - m - f(\sec Z - \sec z), \quad (3)$$

or

$$\Delta m = M - m - fF, \quad (4)$$

where

$$F = \sec Z - \sec z. \quad (5)$$

The photometer measures $M - m$ and the quantity Δm , which gives a point on the light-curve of the variable, is easily found from (3) if the value of f is known.

At Cambridge the value of f for a B-type star, for example, on a good night is about 0.4, and from a set of 3 or 4 determinations of $M - m$ the probable error of the mean may be of the order of $\pm 0^m.002$; it is thus clear that if the value of f is known only approximately, the range of values of F must be strictly limited if errors of the order, say, of $0^m.005$ are not to be introduced into the derived values of Δm . For example, if $Z = 10^\circ$ and $z = 15^\circ$ the numerical value of the extinction correction fF is $0^m.008$ if $f = 0.4$, and $0^m.010$ if $f = 0.5$. But if $Z = 40^\circ$ and $z = 45^\circ$, the numerical value of fF is $0^m.044$ if $f = 0.4$ and $0^m.054$ if $f = 0.5$. If the true value of f is 0.45, the error in Δm by taking f to be 0.4 or 0.5 is almost negligible in the first example, whereas in the second it is $0^m.005$. It may be expected, however, that in forming a normal place for a particular phase of the light-curve from observations on several nights the uncertainties due to the use of approximate values of f may tend to cancel out in the mean. According to Stebbins,* his practice is to limit the range of sidereal time within which observations of a variable star are to be made in such a way that the extinction correction fF shall not exceed $0^m.025$, the value of f being estimated. With this procedure the error resulting from an inaccurate value of f can be kept comparatively small. But in many instances it may be desirable—or it may be inevitable, owing to circumstances connected with the choice of the comparison star—that the period of observation on a particular night should cover a larger range than is possible if Stebbins's criterion is to be rigidly adhered to. Hence it is important to have a much more accurate value of the extinction factor f than can be obtained merely by estimating the quality of the seeing. With a fairly accurate knowledge of f there is no reason to suspect extinction corrections of $0^m.050$, or even more, of any serious inaccuracy. The consequence is that the observer has a more extensive area of sky in which to select the most suitable comparison star. The exigencies of the English climate make it equally desirable to us at Cambridge that as little time as possible should be spent on special observations designed to evaluate the extinction factor, and, if such observations can be dispensed with entirely, the whole of the available observing time can be devoted to the

* *Publ. of Washburn Obs.*, 15, 17, 1928.

photometry of the variable stars on the programme or any other investigations of a like character.* In this paper methods of deriving the factor f from the photometric readings made on the comparison stars will be demonstrated. A comparison star will thus play two parts; it will serve as the photometric standard of reference for the variable star under observation, and it will also provide information from which the values of f can be obtained throughout the period of observation.

3. The general formula for the effect of atmospherical extinction, applicable to zenith distances of about 75° , is

$$m - m_0 = f \sec z - g \sec^3 z, \quad (6)$$

which is derived as follows:—

Consider the atmosphere stratified in thin spherical layers (fig. 1) and

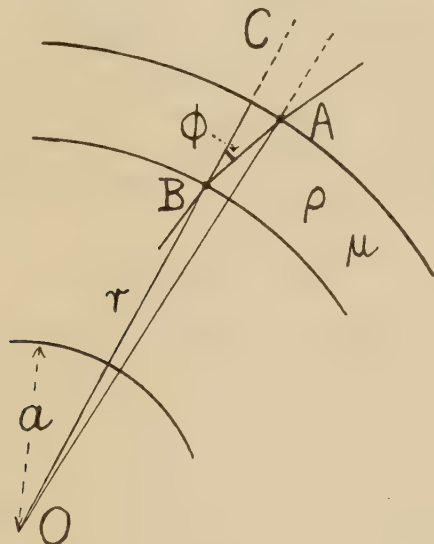


FIG. 1.

regard the density ρ and the index of refraction μ constant in any layer. Let AB , in a particular layer, be the path followed by a stellar ray which eventually reaches the observer. Let I be the intensity of brightness of the star as seen from A and $I - dI$ the intensity at B . The proportionate diminution in intensity varies as the density and as the length of the path traversed; hence we can write

$$\frac{dI}{I} = k' \rho AB, \quad (7)$$

where k' is a constant. If r is the distance of B from O , the Earth's centre, and ϕ is the angle ABC , we have †

$$\mu r \sin \phi = \mu_0 a \sin z, \quad (8)$$

* For a brief account of the Cambridge photometer, see *Observatory Magazine*, 66, 76, 1933, and *M.N.*, 93, 441, 1933.

† See, for example, the author's *Spherical Astronomy*, p. 63.

where μ_0 is the index of refraction of the air at the Earth's surface, a is the Earth's radius and z is the star's observed zenith distance. If dr is the difference between the geocentric distances of A and B ,

$$AB = dr \sec \phi.$$

Hence from (7) and (8)

$$\frac{dI}{I} = \frac{k' \rho \mu r dr}{(\mu^2 r^2 - \mu_0^2 a^2 \sin^2 z)^{1/2}}. \quad (9)$$

Put

$$\mu = \mu_0(1+n) \quad \text{and} \quad r = a(1+s).$$

Here n and s are small quantities; the former varies between 0 and -0.00029 and the latter between 0 and $1/40$ if we assume that the effective height of the atmosphere is about 100 miles. We then have *

$$\log \frac{I}{C} = ak' \log e \int_{s_0}^s \frac{\rho(1+n)(1+s)ds}{\{(1+n)^2(1+s)^2 - \sin^2 z\}^{1/2}},$$

in which C is a constant of integration and s_0 is the value of s at the upper level of the atmosphere. If I_0 denotes the true intensity (as measured outside the atmosphere) and I the observed intensity at the Earth's surface, we obtain, writing k for $k' \log e$,

$$\log \frac{I_0}{I} = ka \int_0^{s_0} \frac{\rho(1+n+s)ds}{\{\cos^2 z + 2(n+s)\}^{1/2}}, \quad (10)$$

where we keep only the first-order terms in n and s in the numerator and denominator of the integrand. If z is not greater than 75° , (10) can be written with sufficient accuracy,

$$\log \frac{I_0}{I} = ka \sec z \int_0^{s_0} \rho ds - ka \sec z \tan^2 z \int_0^{s_0} \rho(n+s)ds. \quad (11)$$

The first term on the right of (11) is $Mk \sec z$ where M is the mass of a vertical column of air on a base of unit area.

Also,

$$\mu = 1 + c\rho \quad \text{and} \quad \mu_0 = 1 + c\rho_0,$$

where c is a constant, its value for air being 0.226. Hence

$$\begin{aligned} a \int_0^{s_0} \rho(n+s)ds &= \frac{ac}{\mu_0} \int_0^{s_0} \rho(\rho - \rho_0)ds + a \int_0^{s_0} \rho s ds \\ &= \frac{ac}{\mu_0} \int_0^{s_0} \rho^2 ds - \frac{c\rho_0 M}{\mu_0} + a \int_0^{s_0} \rho s ds. \end{aligned} \quad (12)$$

If $a\bar{s}$ is the height of the centre of gravity of a vertical column of air above the Earth's surface, the last term of (12) is $M\bar{s}$. It does not seem possible to give a simple physical interpretation of the first term on the right of (12). But

$$a \int_0^{s_0} \rho^2 ds < a\rho_0 \int_0^{s_0} \rho ds < M\rho_0,$$

* The logarithms of all functions are to base 10 throughout.

and we can write

$$a \int_0^{s_0} \rho^2 ds = \kappa M \rho_0,$$

where κ is positive and less than unity.

We then obtain from (11)

$$\log \frac{I_0}{I} = kM \sec z - kM \sec z \tan^2 z \left(\frac{\kappa c \rho_0}{\mu_0} - \frac{c \rho_0}{\mu_0} + \bar{s} \right)$$

or

$$\log \frac{I_0}{I} = A \sec z - B \sec z \tan^2 z. \quad (13)$$

Now $c \rho_0 / \mu_0 = 0.0003$, and it is hardly likely that the value of $\bar{s}$ can exceed 0.005 (corresponding to a height of about 20 miles for the centre of gravity). Thus $B/A < 0.005$. As an estimate, the value of B/A may be taken to be 0.003. We can thus write (13) with sufficient accuracy in the form

$$\log \frac{I_0}{I} = A \sec z - 0.003 A \sec^3 z. \quad (14)$$

If m and m_0 are the stellar magnitudes corresponding to I and I_0 , we have from (14)

$$m - m_0 = f \sec z - 0.003 f \sec^3 z, \quad (15)$$

where $f = \frac{5}{2} A$.

4. If the difference of magnitude between a variable star and the comparison star is being measured at large zenith distances, the formula corresponding to (3) is

$$\Delta m = M - m - f(\sec Z - \sec z) + 0.003 f (\sec^3 Z - \sec^3 z). \quad (16)$$

To obtain some idea of the numerical value of the second term in (16) we shall take $f = 0.4$.

As an extreme case, let $Z = 71^\circ$, $z = 70^\circ$ (in the latitude of Cambridge this corresponds to meridian observations of stars of declination -19°); then

$$\Delta m = M - m - 0^m.060 + 0^m.004.$$

For $Z = 62^\circ$, $z = 60^\circ$,

$$\Delta m = M - m - 0^m.052 + 0^m.002.$$

For $Z = 50^\circ$, $z = 45^\circ$,

$$\Delta m = M - m - 0^m.057 + 0^m.001.$$

These examples serve to show that if comparison stars can be chosen to satisfy $Z - z < 1^\circ$, 2° and 5° for $Z = 71^\circ$, 62° and 50° the simple formula (3) is adequate even at these large zenith distances. If these conditions cannot be approximately satisfied, (16) must be used and a more accurate value of the numerical coefficient of the second term (*i.e.* the ratio $B:A$) must be found by observation.

The foregoing is not intended in any way to suggest that observations of variable stars which transit at small or moderate zenith distances should be continued up to zenith distances approaching 70° —in any event, the

photometric readings increase too rapidly with zenith distance beyond 45° to allow accurate determinations of Δm —but to suggest that the extension of an observing programme to stars which transit at rather high zenith distances ought, under good conditions, to be entirely practicable. For stars coming under the latter category, the period of observation will necessarily be restricted to about one hour or so on either side of the meridian.

5. The usual method of determining the extinction factor f is by means of two distinct sets of observations of a pair of stars, separated by a suitable time interval. If Δm_1 denotes the observed difference of magnitude of the two stars at the earlier epoch and Δm_2 at the second epoch, we have by (3)

$$\Delta m = \Delta m_1 - fF_1 \quad (17)$$

and

$$\Delta m = \Delta m_2 - fF_2, \quad (18)$$

in which F_1 and F_2 are the values of F at the two epochs and Δm , as before, is the difference of the true magnitudes of the stars. Eliminating Δm we have

$$f = \frac{\Delta m_2 - \Delta m_1}{F_2 - F_1}. \quad (19)$$

If f is to be found with reasonable accuracy, the interval should be such as to make $F_2 - F_1$ about 0.4 or 0.5 at least. With two suitably chosen stars this interval need not exceed two hours. When f has been determined by means of (19), the difference of the true magnitudes of the stars, Δm , can be found from (17) or (18). Observations on several nights will furnish as many values of Δm , which, of course, all ought to be identical. The mean of half a dozen results should have a probable error not exceeding $\pm 0^m.005$. We can therefore regard Δm as known with sufficient accuracy, and on any subsequent night it is only necessary to make one set of observations so that f can be found from (17) alone; it is essential, of course, that the observations be made at a time when the value of F_1 is numerically large, say 0.3.

In this procedure it has been tacitly assumed that the state of the sky at the earlier epoch is the same as at the later epoch, whatever the fluctuations, if any, in the interval may be. If the seeing is fairly uniform, the value of f found from (19) may be regarded as a *representative* value for the period of observation concerned. It may easily be seen from (17) and (18) that if f_1 and f_2 are the extinction factors at the first and second epochs of observation, the value of f derived from (19) is the *mean* value of f_1 and f_2 only if $F_1 = -F_2$. In other cases the representative value of f may differ appreciably from the mean of f_1 and f_2 .

6. I now describe an absolute method of finding the extinction factor from observations of a single star. The method of the preceding section required special observations; the present method can be based on observations of the comparison star made in the course of ordinary work on the variable concerned, provided the zenith distance of the comparison star is about 40° or so and altering fairly rapidly. For example, δ Persei is used

as the comparison star in our observations of Algol, and the extinction correction fF on an average night is at most $0^m.010$ for hour-angles between 19^h and 1^h . Thus observations can be made over a considerable part of the night in autumn with small possibility of error in the extinction correction fF even if f is only approximately known. The photometer readings when the hour-angle of δ Persei is about 19^h or 20^h can be picked out for the purpose of the method.

If T is the number of seconds taken by the electrometer needle to move over a given number of divisions on the photometer scale when the star is under observation, and T_0 the number corresponding to a similar observation supposing the effect of the atmosphere to be eliminated, we have

$$\frac{I_0}{I} = \frac{T}{T_0}.$$

Hence (14) can be written, the small term being omitted,

$$\log \frac{T}{T_0} = A \sec z,$$

or, since $A = \frac{2}{5} f$,

$$\log \frac{T}{T_0} = \frac{2}{5} f \sec z. \quad (20)$$

If $T + dT$ is the observed time when the zenith distance is $z + dz$, we have

$$\log e \cdot \frac{dT}{T} = \frac{2}{5} f \sec z \tan z \, dz.$$

If dz is expressed in degrees, this becomes

$$\frac{dT}{T} = 0.01607 f \sec z \tan z \, dz. \quad (21)$$

An interval of about 10 minutes is generally necessary so as to make dz a reasonably large quantity. dT/T is measured accurately by the photometer, and hence f can be calculated from (21). However, in this form (21) is not very satisfactory unless dz is found with fair accuracy, and this generally means special calculations. A more suitable form is found as follows:—

If ϕ is the latitude and δ and H the declination and hour-angle of the star respectively,

$$\cos z = \sin \phi \sin \delta + \cos \phi \cos \delta \cos H,$$

whence

$$\sin z \, dz = \cos \phi \cos \delta \sin H \cdot dH.$$

If dz is expressed in degrees and dH in minutes of time, this becomes

$$\sin z \, dz = \frac{1}{4} \cos \phi \cos \delta \sin H \cdot dH.$$

Hence (21) becomes

$$\frac{dT}{T} = 0.00402 f \sec^2 z \cos \phi \cos \delta \sin H \cdot dH,$$

and, putting

$$\frac{1}{G} = 0.00402 f \sec^2 z \cos \phi \cos \delta \sin H,$$

we obtain

$$f = \frac{G}{T} \cdot \frac{dT}{dH}. \quad (22)$$

As the times at which the observations are made are recorded, dH is found at once. The values of z for different hour-angles have to be found in any event—for the calculation of F —so no extra calculation is necessary as regards z . A table giving G at intervals of 30^m in H , say from $H=3^h$ to $H=6^h$, can be computed in a few minutes. It is much more convenient to use the data for G in the form of a graph from which the appropriate value of G can be found at once. The following table gives the values of G for δ Persei:—

TABLE I

Values of G

	^h	^h	^h	^h	^h	^h	^h
H	3.0	3.5	4.0	4.5	5.0	5.5	6.0
G	653	530	435	357	295	247	208

The method depends for success on two factors: (a) the uniform performance of the photometer over the period of observation, and (b) the constancy of the seeing throughout the same interval. As regards (a) there is little reason to suspect any variation in the sensitivity of the instrument over such a short interval as required in this particular set of observations; actually there is no evident variation over intervals of several weeks, provided the applied voltage on the photo-electric cell remains constant. That leaves (b) as the effective factor.

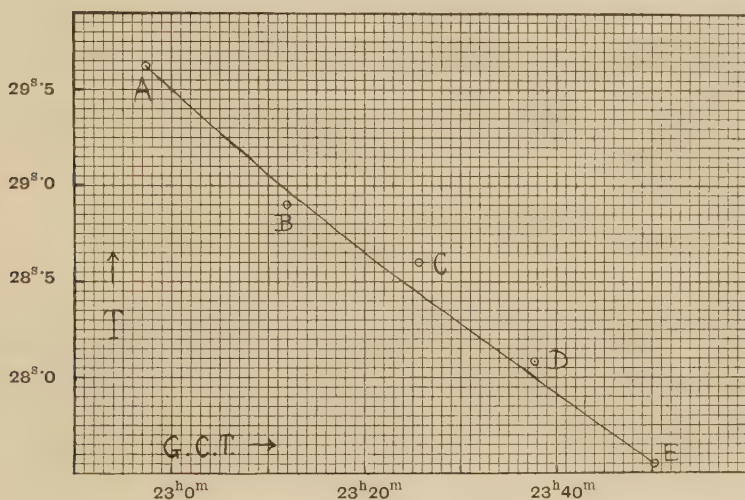
The observations on which formula (22) was first tried were made on 1933 September 10. The seeing was excellent and steady, and the extinction factor for Go stars made near the time at which δ Persei was observed was 0.35. As δ Persei is a B5 star, this extinction factor corresponds to about 0.39 for δ Persei. Table II gives the details of the observations.

TABLE II

Observations of δ Persei

Observation	^T (observed)	^H	^T (from graph)	^{dH}	^G	^f
	^s	^h ^m	^s	^m		
A	29.62	5 15 E	29.62	14.5	270	0.42
B	28.90	5 1	28.96	14.0	295	0.38
C	28.60	4 48	28.44	11.8	318	0.42
D	28.08	4 37	28.00	12.7	342	0.44
E	27.55		27.55			

The second column gives the observed times T (each is the mean of 5 or 6 readings). The 3rd column gives the star's hour-angle (E) midway between consecutive observations AB , BC , etc. The values of T are shown plotted in fig. 2, the G.C.T. of the observation being the abscissa. In the 4th column are the values of T taken from the smooth curve shown in the figure. The observation C is out of harmony with the general run of the curve, only to be explained by a temporary deterioration of the seeing round about the time this observation was being made. The 5th column of Table II contains the intervals dH between the consecutive observations and the 6th column contains the values of G taken from a curve based on the data of Table I. In the last column are to be found the values of f calculated by means of

FIG. 2.—Observations of δ Persei.

formula (22). The mean is 0.41, in good agreement with the value found by a different method.

It must be confessed, however, that, despite this success of the first attempt to apply formula (22), the application of the method in further trials did not always lead to dependable results. Failures are to be attributed to variations in the quality of the seeing during the period of observation; that there are variations is easily detected as a rule from the general run of the values of T .

7. A method of wide generality for the determination of the extinction factor will be described in this section. Like the previous method, it can be applied to the readings of any comparison star observed during the ordinary routine of a night's work on variable stars, but, unlike the previous method, it is not dependent on any special condition associated with zenith distance. It is independent of any variation in the state of the seeing during the evening; it has its greatest value, in fact, when the disturbances in the atmosphere (other than those due to clouds) are greatest and most rapid, for the successive values of f can be determined with considerable accuracy and used in calculating the extinction corrections.

The principle of the method depends essentially on the constancy of performance of the photo-electric cell over a considerable interval of time. A good cell can be trusted to fulfil this condition provided the applied voltage does not vary. The Cambridge photometer is based on a "null" principle, and any variations in the electrometer do not affect the results. Thus, if a star is observed on any number of nights at the same zenith distance and under identical conditions of seeing, the times recorded over a given number of scale-divisions of the photometer should all be the same.

If T_1 is the observed time when the extinction factor is f_1 and the zenith distance is z , we have, from (20),

$$\log \frac{T_1}{T_0} = \frac{2}{5} f_1 \sec z. \quad (23)$$

On another night, when the extinction factor is f_2 and the observed time is T_2 , we have for the same zenith distance z ,

$$\log \frac{T_2}{T_0} = \frac{2}{5} f_2 \sec z. \quad (24)$$

Hence from (23) and (24),

$$f_2 = f_1 + \frac{5}{2} \cos z \log \frac{T_2}{T_1}. \quad (25)$$

We shall regard f_1 as the standard value of the extinction factor (in the illustration of the method it will be taken to be 0.4), and associated with this standard f_1 there will be a particular standard T_1 —for a given star—for a particular value of z . These standards being known, f_2 can be readily found by means of (25). In Table III the values of $(f_2 - f_1)$ can be read at a glance for any value of z and T_2/T_1 covered by the table.

TABLE III
Values of $f_2 - f_1$

z	T_2/T_1							
	0.90	1.0	1.1	1.2	1.3	1.4	1.5	1.6
0								
0	-0.10	0	+0.10	+0.20	+0.28	+0.36	+0.44	+0.51
10	-0.10	0	0.10	0.20	0.28	0.36	0.43	0.50
20	-0.10	0	0.10	0.19	0.27	0.34	0.41	0.48
25	-0.09	0	0.09	0.18	0.26	0.33	0.40	0.46
30	-0.09	0	0.09	0.17	0.25	0.32	0.38	0.44
35	-0.08	0	0.08	0.16	0.23	0.30	0.36	0.42
40	-0.08	0	0.08	0.15	0.22	0.28	0.34	0.39
45	-0.07	0	0.07	0.14	0.20	0.26	0.31	0.36
50	-0.07	0	0.07	0.13	0.18	0.23	0.28	0.33
55	-0.06	0	0.06	0.11	0.16	0.21	0.25	0.29
60	-0.05	0	0.05	0.10	0.14	0.18	0.22	0.25

The photo-electric cell in use is kept charged at 180 volts with a possible variation of 1 volt either way. The sensitivity factor for an increase of 1 volt is 1.04. Thus if the voltage is 181, the observed times T can all be corrected to the standard voltage 180. As the voltmeter reads to 1 volt only,

there is consequently a possibility of an error of half a volt ; this will lead to an error of about 0.02 in the value of $f_2 - f_1$, as is evident from Table III.

8. The application of this method will now be illustrated from a series of observations of η Pegasi and β Draconis (both Go stars) on 13 nights between 1933 August 23 and September 20, undertaken for the definite purpose of evaluating the extinction factor by means of formula (19). The intervals between the first and second epochs of observation varied from 2 to 2½ hours. On the first and last nights the cell voltage was 179 and on all the others 180. The photometer times T on the former nights were reduced to voltage 180, and these values are given in the subsequent tables. On four nights of good seeing, the mean value of f was found to be close to 0.4, and the observations were used to determine the standard f_1 of formula (25). The details of the observations are given in Table IV, the first half of the table referring to the earlier epoch of observation and the second half to the later epoch. The "mean" values of f found by means of (19) are shown in column 6 and again in the last column.

TABLE IV
Observations of η Pegasi and β Draconis

Date	η Pegasi		β Draconis		"Mean" f	η Pegasi		β Draconis		"Mean" f
	Zenith Dis- tance	T_1	Zenith Dis- tance	T_3		Zenith Dis- tance	T_2	Zenith Dis- tance	T_4	
Aug. 23	°	^s	°	^s		°	^s	°	^s	
Sept. 10	44.3	25.9	15.1	21.5	0.39	25.0	22.6	37.1	22.6	0.39
	44.8	24.5	14.7	20.2	0.39	24.1	24.7	38.7	24.7	0.39
					(0.35)					(0.44)
19	42.3	26.5	17.3	22.5	0.37	24.2	22.9	38.4	23.1	0.37
20	43.0	26.4	16.5	22.2	0.41	26.4	22.5	35.0	22.3	0.41
Mean	43.6	25.8	15.9	21.6	0.39	24.9	23.2	37.3	23.2	0.39
					(0.38)					(0.40)

It will be noticed that on September 10 the values of T_1 and T_3 are distinctly below the corresponding means at the first epoch, and that the values of T_2 and T_4 are distinctly above the means at the second epoch. This is due, as will be seen from the results of Table VI, to the fact that at the earlier epoch on this date the seeing was distinctly better than at the second epoch. A preliminary calculation shows that corresponding to T_1 and T_3 on September 10 the value of f is 0.35, and corresponding to T_2 and T_4 the value of f is 0.44. These values are shown in brackets in the table, and are used in forming the means on which the subsequent investigation is based. From the last line of Table IV we conclude that the standard performance of the photometer as regards η Pegasi is represented by the following results :—

- (a) $T = 25^s.8$ for Z.D. $43^{\circ}.6$ when $f = 0.38$,
- (b) $T = 23^s.2$ for Z.D. $24^{\circ}.9$ when $f = 0.40$.

These results should, of course, be reconcilable. We reduce them both, first of all, to zenith distance 0° . If T' is the time corresponding to zenith distance 0° ,

$$\log \frac{T}{T'} = \frac{2}{5} f (\sec z - 1). \quad (26)$$

We find

$$\begin{aligned} (a) \quad T' &= 22^s.6 \text{ for } f = 0.38, \\ (b) \quad T' &= 22^s.4 \text{ for } f = 0.40. \end{aligned}$$

Reducing the first to $f = 0.40$ by means of Table III we obtain $T' = 23^s.0$. The two values, $23^s.0$ and $22^s.4$, are in fairly good agreement. Taking the mean, we regard

$$T' = 22^s.7 \text{ (at the zenith)}$$

as the standard performance of the photometer for η Pegasi under the standard conditions of seeing given by $f = 0.40$.

In a similar manner we find for β Draconis that

$$T' = 21^s.3 \text{ for } f = 0.38 \text{ and } 21^s.1 \text{ for } f = 0.40,$$

and using Table III we obtain, for $f = 0.40$, the values $21^s.7$ and $21^s.1$. Taking the mean, we regard

$$T' = 21^s.4 \text{ (at the zenith)}$$

as the standard performance of the photometer for β Draconis under the standard conditions of seeing given by $f = 0.40$.

If Δm is the difference of the true magnitudes of the two stars we have

$$\begin{aligned} \Delta m &= \frac{5}{2} \log \frac{22.7}{21.4} \\ &= 0^m.064. \end{aligned}$$

The mean magnitude found by the method of section 5 from the observations on the 13 nights is $0^m.061 \pm 0^m.004$. The two results are in remarkably good agreement.

9. Consider now the individual observations of η Pegasi made on the 13 nights. With $T' = 22^s.7$ we can find the value of T , corresponding to zenith distance z and under standard conditions from (26), which is

$$\log \frac{T}{T'} = \frac{2}{5} f (\sec z - 1),$$

where $f = 0.4$. These values of T can be most conveniently exhibited diagrammatically as in fig. 3, in which the curve for β Draconis is also shown. For any observation we find from the curve the value of T corresponding to the particular zenith distance. If we denote this standard value of T by T_1 and the observed value by T_2 , the value of f_2 at the time of observation is found immediately from Table III. In the same way the value of f_2 can be deduced from the corresponding observation of β Draconis. These two values should, of course, be identical, for they refer to the same instant.

Moreover, they are deduced from entirely independent data when once the standard performance of the instrument has been determined for each star.

In Table V the second and third columns give the zenith distances of η Pegasi and β Draconis, the upper line referring to the earlier epoch of observation and the lower line to the second epoch. The fourth and fifth

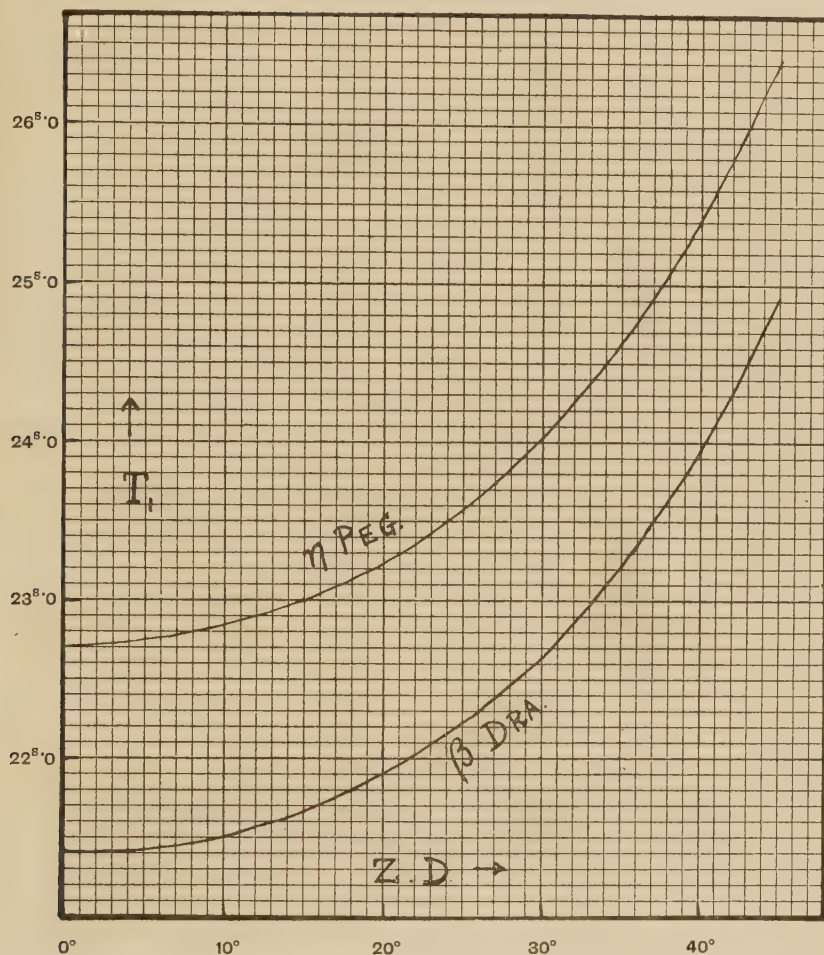


FIG. 3.—Standard Values of T_1 for η Pegasi and β Draconis.

columns give the *observed* difference of magnitude at the first and second epoch, and the next two columns give the corresponding values of F computed by formula (5). In the penultimate column are the “mean” values of f obtained by means of formula (19). The last column gives the difference of true magnitude of the two stars, found by means of (17) or (18).

Giving half-weight to the observations on August 26 and 30, when the sky was particularly “thick” and really unsuitable for photometric work, the mean value of Δm is

$$0^{\text{m}}.061 \pm 0^{\text{m}}.004.$$

TABLE V
Observations of η Pegasi and β Draconis

Date (1933)	Zenith Distance		Δm_1 (in units of 0 ^m .001)	Δm_2	F_1	F_2	"Mean" f	Δm
	η Peg.	β Dra.						
Aug. 23	°	°			+	-		^m
	44.3	15.1	+202	- 2	0.363	0.151	0.39	0.061
	25.0	37.1						
25	39.6	20.2	+208	-28	0.229	0.173	0.59	73
	24.4	38.1						
26	44.3	15.2	+399	+50	0.362	0.039	0.87	85
	28.6	31.9						
28	39.5	20.1	+177	-17	0.230	0.149	0.51	60
	25.0	37.0						
30	38.5	21.1	+265	-32	0.205	0.104	0.96	68
	26.4	35.0						
Sept. 4	39.4	20.2	+176	-58	0.228	0.204	0.50	62
	23.1	41.1						
5	41.1	18.5	+180	-43	0.273	0.234	0.44	60
	23.2	40.8						
8	41.4	18.1	+190	-62	0.282	0.227	0.50	49
	23.3	40.5						
10	44.8	14.7	+207	-10	0.376	0.185	0.39	60
	24.1	38.7						
14	43.2	16.5	+199	-22	0.369	0.286	0.44	55
	24.7	38.0						
15	44.3	15.5	+284	-22	0.356	0.133	0.62	62
	25.5	36.3						
19	42.2	17.3	+169	- 7	0.303	0.180	0.37	59
	24.2	38.4						
20	43.0	16.5	+189	+13	0.324	0.103	0.41	55
	26.4	35.0						

In Table VI the second column gives the standard times T_1 , derived from fig. 3, corresponding to the several zenith distances of η Pegasi, the upper line referring to the earlier epoch of observation and the lower to the second epoch. The observed times T_2 are given in the third column and the ratios T_2/T_1 in the fourth. The next three columns give similar data for β Draconis. In the next column the "mean" values of f , derived by means of (19), are reproduced. In the last two columns are the values of f found by means of Table III.

Throughout, the values of f in the last two columns are practically identical for the two stars at each epoch, and on the great majority of the nights the true average of f corresponds closely to the "mean" or representative value of f determined from (19). As explained in section 5, this latter value is not equivalent to the true mean unless the seeing is constant or $F_1 = -F_2$.

The conclusion is that if it is necessary to determine f with considerable accuracy, say correct to 0.02, the method described will achieve the purpose adequately. Further, the application of the method is extremely simple,

TABLE VI
Determination of f

Date (1933)	η Pegasi			β Draconis			"Mean" f	f	
	T_1 (stand- ard)	T_2 (ob- served)	$\frac{T_2}{T_1}$	T_1 (stand- ard)	T_2 (ob- served)	$\frac{T_2}{T_1}$		η Peg.	β Dra.
	s	s		s	s				
Aug. 23	26.2	25.9	0.99	21.7	21.5	0.99	0.39	0.38	0.39
	23.6	22.6	0.96	23.5	22.6	0.96		0.36	0.37
25	25.3	28.7	1.13	21.9	23.7	1.08	0.59	0.50	0.48
	23.5	23.5	1.00	23.7	24.1	1.02		0.40	0.42
26	26.2	54.8	2.09	21.7	38.0	1.77	0.87	0.97	0.99
	23.9	37.2	1.56	22.8	35.5	1.55		0.83	0.81
28	25.3	25.3	1.00	21.9	21.5	0.98	0.51	0.40	0.38
	23.6	24.5	1.04	23.5	25.0	1.06		0.44	0.45
30	25.2	40.1	1.60	22.0	31.4	1.43	0.96	0.80	0.76
	23.7	37.1	1.57	23.2	38.2	1.65		0.84	0.84
Sept. 4	25.3	29.8	1.18	21.9	25.3	1.16	0.50	0.54	0.55
	23.4	24.0	1.03	24.1	25.3	1.05		0.43	0.44
5	25.6	24.5	0.96	21.8	20.8	0.96	0.44	0.37	0.36
	23.4	22.5	0.96	24.1	23.4	0.97		0.36	0.38
8	25.7	27.3	1.06	21.8	22.9	1.05	0.50	0.44	0.45
	23.4	26.5	1.13	24.1	28.0	1.16		0.53	0.52
10	26.3	24.5	0.93	21.6	20.2	0.94	0.39	0.35	0.34
	23.5	24.7	1.05	23.8	24.7	1.04		0.45	0.43
14	26.0	27.6	1.07	21.7	23.0	1.06	0.44	0.45	0.46
	23.5	28.0	1.19	23.7	28.6	1.21		0.58	0.56
15	26.2	39.0	1.49	21.7	30.0	1.38	0.62	0.71	0.73
	23.6	29.6	1.25	23.4	30.2	1.29		0.62	0.61
19	25.8	26.4	1.02	21.8	22.5	1.03	0.37	0.42	0.43
	23.5	22.5	0.96	23.7	23.1	0.98		0.36	0.38
20	26.0	26.4	1.02	21.7	22.2	1.02	0.41	0.42	0.42
	23.7	22.5	0.95	23.2	22.3	0.94		0.35	0.35

and it is thus easy to keep a check on the state of the seeing at intervals throughout a night's observing.

Summary

10. The paper gives a general discussion of the determination of the atmospherical extinction factor f for use in precise photometric observations.

In section 3 a general formula for the extinction effect on the magnitude of a star is derived.

Two methods for finding the value of f accurately are given and illustrated by means of recent observations.

Observatory, Cambridge :
1933 November 27.

TOTAL SOLAR ECLIPSE OF 1932 AUGUST 31 :
REPORT OF MCGILL EXPEDITION.

J. S. Foster, Ph.D.

The McGill attempt to photograph the flash spectrum and the corona at the 1932 total solar eclipse arose naturally from a favourable set of circumstances. In the first place, the region of totality included the Macdonald Physics Laboratory. Still more fortunately, our spectroscopic apparatus was well suited to this work and could be placed in the hands of an enthusiastic group experienced in its use.

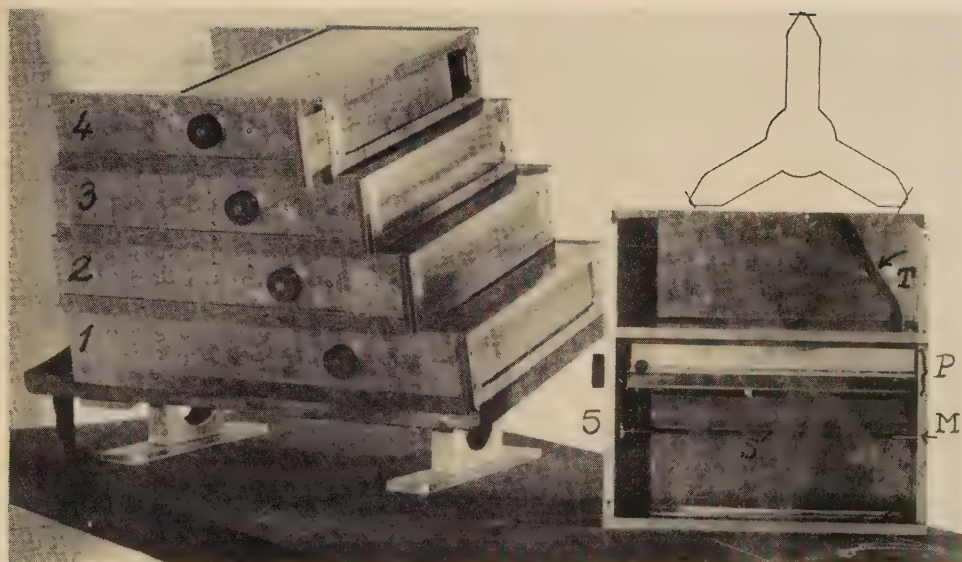
The main interest centred in the adjustment and use of a large concave grating set in a new mounting, and located at Magog. It is now well known that clouds prevented good observations at this point ; hence this report is written mainly for the interest there may be in the optical arrangement.

The grating was ruled by Professor Wood and Mr. H. M. O'Bryan on a Bausch & Lomb blank having a radius of curvature of 30 feet. Fifteen thousand lines per inch were ruled on an area $5\frac{1}{2}$ by 3 inches. The outstanding characteristic of this freak grating is its amazingly weak central image. Owing to this rare feature, one may say that the exceptional intensity found in the first order spectrum on one side is not obtained at the expense of higher orders, but from the central image. As a matter of fact, the second order has far more than average intensity.

In our usual Wadsworth mounting, the axis of the grating intersects the flat plate on which a region of 1000 angstrom units is recorded in good focus with a dispersion of a little less than 4 Å. per millimetre. In some researches it is possible and entirely satisfactory to double the range on a single plate by giving the glass a slight curvature. The outer regions of the plate then register less perfect stigmatism. In fact, we find that they do not give first quality analyses of Stark-Lo Surdo effects in the laboratory ; hence this extension was not considered in the proposed analysis of the flash spectrum.

In order to give some idea of the intensity of the first order spectrum produced by this spectrograph, I may say that we have found all other large gratings almost useless in studies of the Stark effect. Since some of the gratings tried were of exceptional quality, and known to produce bright spectra, I conclude that the present instrument brings to the first order from five to ten times as much light as is commonly found there.

For the eclipse programme we used five incident beams instead of one.



In the above plate, each of the parts 1, 2, 3, 4, 5, is a complete camera. The light enters at the front (the slide is partly drawn in one of the units) and is focussed on the plate, P. A cross-section of one of the plate-holders is shown in the inserted ink drawing. The top slide of camera No. 5 is drawn to reveal a ribbon mirror, M, which reflects a narrow band of light through a long slit in the slide, S'. This light is focussed on a horizontal photographic plate held with clasps on a movable brass platform. The latter is guided at one of its ends by slip rings round a steel rod, and is moved by means of a linen tape, T.

These were so directed as to allow the spectra to come at different vertical heights on plates arranged near the grating axis. With angles of incidence chosen to bring into the camera successive regions of 1000 Å. each, a given plate needed to be stepped back about 2 inches from its "shorter wave-length" neighbour.

The explanation of this lies in the relation between the focal distance, d , and the angle of incidence, i ,

$$d = R/(1 + \cos i),$$

where R is the radius of curvature of the grating. The individual units in the camera (Plate 2) were therefore built in a compact stairway formation, and covered a region of 5000 Å.

The basis of each plate holder is an aluminium casting 12 inches long in the form of a central shaft with three heavy fins projecting at angles of 120° , as shown at P in the photograph of Plate 2. In the same place the reader will find a cross-section of the holder sketched in ink. Brass ends of the axle project into the walls of the camera box with enough play to allow easy removal for the purpose of loading. An external knob is fastened to each holder by a screw which passes loosely through the box and into the brass axle. Inside, a very short section of the latter is in the form of a triangular prism, and against this a stiff spring presses. The arrangement is such that as one turns the knob, the stable positions are those in which one of the photographic plates is perpendicular to the incident light. By turning the five knobs in succession, the operator is able to change all plates in a few seconds.

A set of moving plates was also included on the following plan. About an inch in front of each stationary plate a mirror $\frac{1}{8}$ inch wide was tilted to throw a narrow strip of the spectrum through a $\frac{1}{16}$ -inch slit, and allow it to focus on a horizontal moving plate which was operated by a motor at the rear of the camera. Mirrors satisfactory for this purpose were made by Professor Harkness. A dozen selected glass rods were placed side by side and mounted on a solid base with wax. They were ground and polished until what may be described as a second quality optical surface was obtained. After silvering the best five were selected. Metal caps were cemented on the ends of the mirrors which were then mounted with a minimum of strain. Even so, the weight of the rod itself slightly distorted the plane surface. Since we are interested only in the relative intensities of lines on the moving plates, and not in absolute wave-lengths, this distortion is of little importance.

The introduction of the narrow mirrors obviously means that a band of light $\frac{1}{8}$ inch wide will be missing from the spectrum of crescents on the vertical plates. This is a small fraction of the spectrum, since the full crescents are 1.6 inches high. Moreover, it is easy to so adjust the mirrors reflecting the incident beams to the grating as to allow the portion falling on the ribbon mirror to be slightly off the axis of the spectrum. There is then at least even chances that the mirrors will cause no loss whatever.

A remaining feature of the spectrograph is more especially concerned with the way in which the light was handled from the source. The 12-inch

mirror was removed from the Hills cœlostæt, kindly lent by Professor Stratton, and in its place five 6-inch mirrors were mounted at slightly different angles. These mirrors reflected the light to a second set of five plane mirrors (25 feet distant) and thence to the grating.

For a special purpose, the treatment of one of the beams was varied from the above. In this case the light from the cœlostæt went first to a concave mirror which formed an image of the Sun at a slit placed part way back toward the cœlostæt, but outside the range of the other beams. From the slit the light proceeded to a second concave mirror and back in a parallel beam to the plane mirror which fed the grating. The slit and second concave mirror were rigidly mounted as a unit, and in preliminary work were used with an arc to give a truly parallel beam essential to the focussing programme.

The focal length of each concave mirror is $10\frac{1}{2}$ feet. The first is of glass made and silvered by Bausch & Lomb specially for this work. The second is the regular speculum mirror used in the laboratory as part of the grating spectrograph.

The above slit was introduced into one unit of an otherwise objective spectrograph for the purpose of comparing the flash spectrum intensities of the ionized calcium lines H and K ($\lambda\lambda$ 3934, 3968) with an infra-red calcium triplet $\lambda\lambda$ 8498, 8542, 8662. This problem was suggested to the writer by Professor Stratton, whose equipment lacked the necessary mirrors. It was, however, part of the Greenwich programme, and has been discussed elsewhere by Davidson.* The H and K lines appeared in second order near the first order triplet. The mirror system permitted good focus, while the many reflections failed to affect seriously the extraordinarily brilliant spectrum.

The instrument was set on cement pillars, with the grating lines and camera box inclined at an angle of 15° to bring the crescents of the flash spectrum into an "upright" position on the plate. A dark room was built round the camera to allow easy sensitizing and loading of plates. A large shutter was constructed, and used to intercept the five beams at a point 3 feet from the grating. Each single stroke of a lever at the side of the shutter released a 2.5-inch roller for one revolution, which was enough to complete one operation.

The range from 10,000 Å. to 7000 Å. was covered in a very satisfactory way, but owing to imperfect mirrors the two remaining plates did not show first-class definition. It will be noticed that green and blue regions were not represented on the programme for this instrument. They were covered by prism instruments of high dispersion located in Montreal, and described below. Excellent plates, sensitive for each region, were supplied by Dr. Mees.

Some idea of the relative positions of the optical parts may be gained from fig. 1, where the cement supports are sketched to aid the perspective, but details of the mountings are omitted. The plane through the axis of the camera box and perpendicular to the grating lines passes through the

* C. Davidson, *M.N.*, 88, 30, 1927.

central mirror of the group near the dark room. Since the other mirrors are distributed either above or below this plane, the corresponding spectra at the camera are respectively below or above the central unit of the camera. Of course the mirror positions are correctly chosen to feed the different camera units with the desired ranges of the spectrum. To avoid needless confusion, some of the beams of light are not represented in the figure.

I am very grateful for a grant from the National Research Council, U.S.A., which paid for the necessary modification in the mounting and for part of

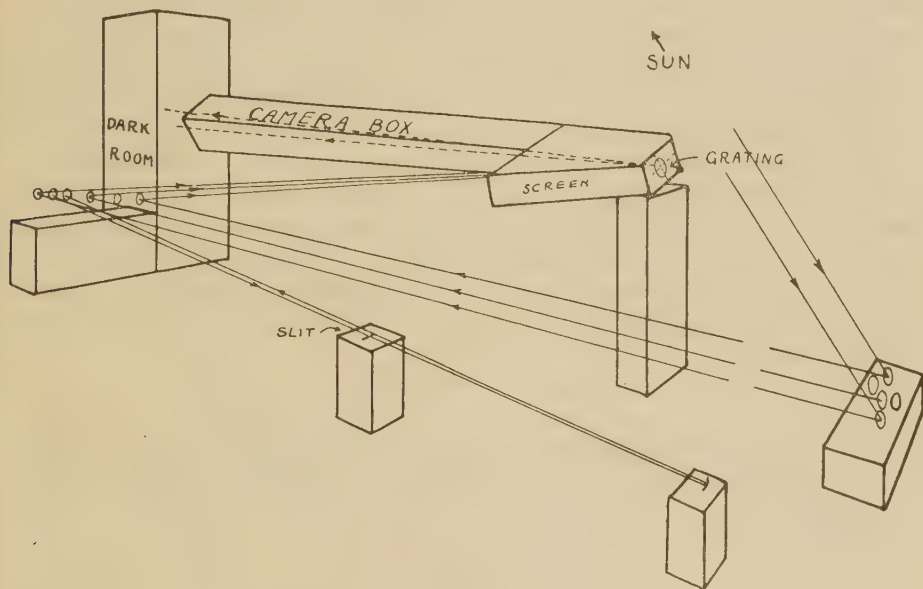


FIG. 1.

the field expenses. The metal camera and mountings were skilfully constructed under the direction of Mr. H. T. Pye.

In addition to the above, our Magog equipment included a fast two-prism glass spectrograph with Zeiss F/4.5 Tessar lenses 3 inches in diameter. This was fed by a heliostat kindly supplied by the Dominion Observatory. The beam was passed through a Zeiss 4.5 lens of 30 cm. focal length to the slit of the spectrograph which was adjusted for an analysis of the spectrum of the corona.

Direct photography of the corona was attempted with a telescopic lens of 5 inches diameter and focal length 5 feet. We are indebted to Professor F. K. Richtmyer of Cornell University for the use of a heliostat which supplied a beam to the telescope and a second beam to a motion picture camera operated by Mr. Jacquemin.

All plates were calibrated in a small Hilger spectrograph by a wedge method.

The construction and care of the spectrographs were in the hands of the writer, together with Professor W. Rowles and Dr. L. Rowles of Macdonald College, and Drs. R. N. H. Haslam and A. H. Snell, National Research

Fellows at McGill. The cœlostat and heliostats at Magog were adjusted by Dr. A. V. Douglas.

The Cambridge Expedition was well established upon our arrival at Magog. Professor Stratton kindly arranged an extension of power lines to supply our tents. We are much indebted to him and to Colonel Waley Cohen.

At Montreal we were fortunate in that the few hours during which a beam of sunlight could be brought into my laboratory included the time of the eclipse. It was therefore possible by means of a heliostat * to reflect a 12-inch beam into the laboratory, and to use two spectrographs near their normal positions. The large McGill six-prism glass spectrograph (with slit) was assigned to the violet region of the flash spectrum where the dispersion was about 2 Å./mm. To obtain the best orientation of the crescent on the slit, an optical system was constructed for rotating the image.† A Hilger E1 quartz spectrograph was focussed with special attention to the Balmer series limit. The heliostat and spectrographs were set up by an independent group consisting of Professor H. W. Harkness (Acadia University), Dr. D. R. McRae (Cambridge), and Dr. R. L. Thornton, National Research Fellow (McGill).

ON ISOTROPIC MODELS OF THE UNIVERSE, WITH SPECIAL REFERENCE TO THE STABILITY OF THE HOMOGENE- OUS AND STATIC STATES.

Herbert Dingle, D.Sc.

1. *Introduction.*—A few months ago I began to consider the possibility of representing the properties of the universe on the large scale, so far as they are known, by a spatially inhomogeneous line-element, the grounds on which homogeneity is generally assumed appearing to be those of convenience rather than probability. In the course of the work it was suggested to me by Dr. R. C. Tolman that an investigation of the stability of spatial homogeneity in the universe would be very interesting, and since equations which I had already arrived at seemed to afford a satisfactory approach to such a problem, the course originally planned became diverted into a consideration of this very pertinent question. It soon became evident that the various questions involved necessitated, first, a general inquiry into the possible forms of a universe assumed (for simplicity in a first investigation) to be isotropic, but not necessarily static or spatially homogeneous, and, secondly, a definition and application of criteria of staticity and spatial homogeneity. The position would then be prepared for a discussion of the stability of these two special conditions.

* We express our best thanks to Dr. D. C. Miller of the Case School of Applied Science for the use of this instrument.

† As designed and used by Professor Dingle, *M.N.*, 93, 340, 1933.

The course of this paper is as follows: We first define staticity and spatial homogeneity in the physical sense, and then deduce criteria for these conditions in terms of the general metric of an isotropic universe. We next derive the most general form of line-element satisfying the assumptions usually made in such investigations, except that of spatial homogeneity, and deduce the particular specialisations of this line-element necessary to make it represent static and spatially homogeneous universes respectively. As an essential part of this inquiry we find the limits of possible transformations of co-ordinates which can be made without violating the conditions assumed. The stability of staticity and spatial homogeneity in the universe is then examined, and a general review is given of the reasons which have led to the general assumption that space is homogeneous. Finally, the physical meaning of some of the results arrived at is briefly discussed.

2. *Definitions of Staticity and Homogeneity.*—According to the general principle of relativity the universe is represented by a four-dimensional Riemannian continuum, of which three dimensions are called “spatial” and are distinguished from the fourth, which is called “temporal,” by the occurrence of a minus sign before the corresponding co-ordinate differentials in the expression for the element of interval. The number of such continua that are theoretically possible is multiply-infinite, and in order to make discussion possible in the almost complete absence of observational knowledge, the assumption was made in the earliest investigations that in the actual universe the physical characteristics of all four-dimensional points were precisely equivalent. This restricted the possible Riemannian manifolds to three, known generally as the Einstein, de Sitter and Euclidean worlds respectively. The two last named, however, are possible geometrical representations of such a universe only because they contain no matter to show differentiating features; the physical characteristics of all four-dimensional points are equivalent because they do not exist. Since the only thing that was known about the universe on the large scale was that it did contain matter, the Einstein world remained as the only one which could represent the universe (except as a possibly close approximation) if it were actually homogeneous in all four dimensions.

The verification of the spectrum phenomenon usually interpreted as a systematic recession of the extra-galactic nebulae, however, showed that if the portion of the universe accessible to observation were typical of the whole, this restriction could no longer be maintained. From our present point of view it is clear that two equally reasonable less stringent restrictions were then naturally open for consideration: all points of space-time could not be regarded as equivalent, but all points of space or all points of time might possibly be so regarded. For some reason the former possibility alone received attention—with, as is well known, considerable success. Possibly this was because the spectrum phenomenon above referred to was looked upon in the first instance as a genuine Doppler effect, in which case different points of time could be distinguished by the densities of matter with which they were associated. This, however, is hardly sufficient justification. What we observe is not a diminishing density of matter, but a displacement

of spectrum lines which might *prima facie* be interpreted as well by the assumption of an unchanging space and an expanding or contracting time as by that of an unchanging time and an expanding or contracting space. This interpretation would, furthermore, avoid what is to many the unsatisfactory conception that we happen only by good fortune to be living at the time when the nebulae are visible. We shall show later, as a matter of fact, that the actual law observed—namely, that the spectrum displacement is proportional to the distance of the nebula—is inconsistent with such an interpretation, but without such a demonstration it would appear premature to neglect this alternative to the current explanation.

We shall call a space of which all points are equivalent a “homogeneous” space, and the space corresponding to a time of which all instants are equivalent a “static” space. The statement of the last paragraph, then, is that a static space might, but does not, permit an explanation of the nebular spectrum phenomenon, while a homogeneous space affords a closely approximate one. The three special cases to which we have referred may then be characterised as follows. A completely homogeneous four-dimensional world is one of which all points of space are alike at all times : a static space is one of which any given point is alike at all times : a homogeneous space is one of which all points are alike at any given instant of time. Clearly a static space is conjugate with a homogeneous time and a homogeneous space with a static time, but we shall limit our attention to space.

We shall characterise space primarily by its physical characteristics—namely, the pressure and density of the matter there, which are potentially observable : the corresponding geometrical qualities follow at once from the general principle of relativity. We accordingly take as a criterion of a static space,

$$\frac{\partial p}{\partial t} = \frac{\partial \rho}{\partial t} = 0, \quad (1)$$

and as a criterion of a homogeneous space,

$$\frac{\partial p}{\partial r} = \frac{\partial \rho}{\partial r} = 0, \quad (2)$$

where the symbols have their usual meanings. In the following paragraphs we shall obtain the most general forms of line-element consistent with these relations and satisfying certain assumptions which will be stated, but it might be well to remark here that a line-element which reduces to that of a homogeneous three-dimensional subspace at each instant of time is not necessarily the line-element of a universe whose space is homogeneous. The points of such a subspace might still be distinguished from one another by time measurements (for instance, the period of a clock at A as observed by B might not be identical with the period of a similar clock at B as observed by A), and the space would not, therefore, satisfy our definition of homogeneity. Since the dimensions of pressure and density include that of time as well as those of space, this is automatically taken account of by our physical criterion.

3. *Simplifying Assumptions regarding the Character of the Universe.*—We shall now seek the line-element of a universe satisfying the assumptions usually made in these investigations except that of spatial homogeneity. Conditions (1) and (2) will then be applied in turn in order to obtain the line-elements of the most general static and homogeneous universes subject to those assumptions.

According to the principle of relativity, any material system can be represented by a line-element of the general form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad (3)$$

where μ and ν have each all values from 1 to 4 inclusive. In our treatment we shall assume that this general form can be simplified as follows:—

(i) We assume that the spatial universe is spherically symmetrical round some point which we may choose as origin of co-ordinates, and that all its metrical and physical properties are isotropic with respect to that point. This implies the possibility of a co-ordinate system in which

$$g_{\mu\nu} = 0 \quad (\mu \neq \nu), \quad (4)$$

and reduces the line-element to the form

$$ds^2 = -F(r, t)[dx^2 + dy^2 + dz^2] + H(r, t)dt^2, \quad (5)$$

where the usual symbols for cartesian co-ordinates and time have been used, and r has its usual meaning. The assumption of physical isotropy gives the further condition,

$$T_1^1 = T_2^2 = T_3^3, \quad (6)$$

where T_μ^ν is the mixed energy-momentum tensor.

(ii) We assume that the universe is such that a system of co-ordinates can be chosen which moves with the matter at every point. This implies that there are no singularities anywhere, except possibly at the origin. The restriction thus imposed on the line-element can be deduced from the condition,

$$T_4^1 = T_1^4 = T_4^2 = T_2^4 = T_4^3 = T_3^4 = 0. \quad (7)$$

(iii) We assume that the material of the universe is subject to no tangential stresses. The condition for this is

$$T_2^1 = T_1^2 = T_3^1 = T_1^3 = T_3^2 = T_2^3 = 0. \quad (8)$$

These assumptions have all a considerable amount of justification. The assumption of isotropy, however, has no greater empirical justification than that of homogeneity which we are discarding. It is necessary to make it, nevertheless, in order that the problem shall remain tractable.

We shall proceed to determine the most general forms of the functions F and H consistent with the conditions (6), (7) and (8), but before doing so we insert a remark on notation and consider what liberty in transforming co-ordinates remains to us without violating the conditions we have assumed.

4. *Remark on Notation.*—We shall find it convenient to use as the spatial

co-ordinate, r^2 instead of r . Since only positive values of r are physically significant, this involves no loss of generality. We accordingly write

$$\sigma \equiv r^2 \equiv x^2 + y^2 + z^2. \quad (9)$$

We shall denote functions of both σ and t by capital Latin letters, functions of σ alone by small Greek letters and functions of t alone by small Latin letters. For absolute constants there remain capital Greek letters, and their use will accord with the custom of Tolman in representing the cosmological constant by Λ . The only exceptions to our convention will be the retention of λ , p , ρ and R for wave-length and for the pressure, density and radius of the universe respectively—these symbols being too well established by custom to be changed—and of the usual tensor indices in the next paragraph. With this notation we can simplify our formulæ by omitting the bracketed independent variable, which usually follows the function symbol, on all occasions after its first introduction; thus $F(r, t)$ will henceforward appear as F .

The differentials of the logarithms of F and H will frequently appear in the work, and we shall represent them by the following symbols:—

$$\left. \begin{aligned} \frac{1}{F} \frac{\partial F}{\partial \sigma} &\equiv L; & \frac{1}{F} \frac{\partial F}{\partial t} &\equiv N; \\ \frac{1}{H} \frac{\partial H}{\partial \sigma} &\equiv M; & \frac{1}{H} \frac{\partial H}{\partial t} &\equiv P. \end{aligned} \right\} \quad (10)$$

We may note here the important identities,

$$\left. \begin{aligned} \frac{\partial L}{\partial t} &\equiv \frac{\partial N}{\partial \sigma}; \\ \frac{\partial M}{\partial t} &\equiv \frac{\partial P}{\partial \sigma}; \\ \frac{\partial}{\partial t}(L + M) &\equiv \frac{\partial}{\partial \sigma}(N + P). \end{aligned} \right\} \quad (11)$$

5. *Permissible Changes of Co-ordinates.*—The principle of relativity asserts that any real functional transformation of co-ordinates can be made without affecting the physical properties of the universe expressed by a given line-element. The conditions we have imposed, however, involve not only physical properties of the universe, but the properties of co-ordinate systems also, and it is necessary to determine at the outset what restrictions those conditions place on our liberty of transformation.

Suppose, then, that we have a line-element satisfying conditions (6), (7) and (8); let its co-ordinates be x^a ($a = 1, 2, 3, 4$). We transform to a system $\bar{x}$, and require that the new energy-momentum tensor, $\bar{T}_\mu^\nu$, shall satisfy the similar conditions,

$$T_\mu^\nu = 0 (\mu \neq \nu); \quad (12)$$

$$T_1^1 = T_2^2 = T_3^3. \quad (13)$$

We have—

$$\begin{aligned}
 T_{\mu}{}^{\nu} &= T_{\beta}{}^{\alpha} \frac{\partial \bar{x}^{\nu}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial \bar{x}^{\mu}} \\
 &= T_{\alpha}{}^{\alpha} \frac{\partial \bar{x}^{\nu}}{\partial x^{\alpha}} \frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}} \quad (\text{from (7) and (8)}) \\
 &= T_1{}^1 \frac{\partial \bar{x}^{\nu}}{\partial x^1} \frac{\partial x^1}{\partial \bar{x}^{\mu}} + T_2{}^2 \frac{\partial \bar{x}^{\nu}}{\partial x^2} \frac{\partial x^2}{\partial \bar{x}^{\mu}} + T_3{}^3 \frac{\partial \bar{x}^{\nu}}{\partial x^3} \frac{\partial x^3}{\partial \bar{x}^{\mu}} + T_4{}^4 \frac{\partial \bar{x}^{\nu}}{\partial x^4} \frac{\partial x^4}{\partial \bar{x}^{\mu}} \\
 &= T_1{}^1 \left(\frac{\partial \bar{x}^{\nu}}{\partial x^1} \frac{\partial x^1}{\partial \bar{x}^{\mu}} + \frac{\partial \bar{x}^{\nu}}{\partial x^2} \frac{\partial x^2}{\partial \bar{x}^{\mu}} + \frac{\partial \bar{x}^{\nu}}{\partial x^3} \frac{\partial x^3}{\partial \bar{x}^{\mu}} \right) + T_4{}^4 \frac{\partial \bar{x}^{\nu}}{\partial x^4} \frac{\partial x^4}{\partial \bar{x}^{\mu}} \quad (\text{from (6)}) \\
 &= T_1{}^1 \left(\frac{\partial \bar{x}^{\nu}}{\partial \bar{x}^{\mu}} - \frac{\partial \bar{x}^{\nu}}{\partial x^4} \frac{\partial x^4}{\partial \bar{x}^{\mu}} \right) + T_4{}^4 \frac{\partial \bar{x}^{\nu}}{\partial x^4} \frac{\partial x^4}{\partial \bar{x}^{\mu}} \\
 &= T_1{}^1 \delta_{\mu}{}^{\nu} + (T_4{}^4 - T_1{}^1) \frac{\partial \bar{x}^{\nu}}{\partial x^4} \frac{\partial x^4}{\partial \bar{x}^{\mu}}. \tag{14}
 \end{aligned}$$

We now impose conditions (12) and (13). The former requires that when $\mu \neq \nu$, either $(T_4{}^4 - T_1{}^1)$ or $\frac{\partial \bar{x}^{\nu}}{\partial x^4} \frac{\partial x^4}{\partial \bar{x}^{\mu}}$ shall vanish. The vanishing of $(T_4{}^4 - T_1{}^1)$, however, would imply a relation between pressure and density impossible to any known form of matter or energy; hence

$$\frac{\partial \bar{x}^{\nu}}{\partial x^4} \frac{\partial x^4}{\partial \bar{x}^{\mu}} = 0 \quad (\mu \neq \nu). \tag{15}$$

If, therefore, there is a value of ν for which $\frac{\partial \bar{x}^{\nu}}{\partial x^4} \neq 0$, $\frac{\partial x^4}{\partial \bar{x}^{\mu}}$ must vanish for every other value of μ . Now there must be at least one such value of ν (say α); otherwise the Jacobian of the transformation would vanish. Hence $\frac{\partial x^4}{\partial \bar{x}^{\mu}}$ must vanish except when $\mu = \alpha$. It cannot vanish when $\mu = \alpha$ also,

or again the Jacobian of the transformation would vanish. Hence $\frac{\partial \bar{x}^{\nu}}{\partial x^4} = 0$ for all values of ν except α . Consequently

$$\frac{\partial \bar{x}^{\nu}}{\partial x^4} = 0; \quad \frac{\partial x^4}{\partial \bar{x}^{\mu}} = 0, \quad \text{except when } \nu = \mu = \alpha. \tag{16}$$

Our condition therefore requires that one, and only one, of the new co-ordinates must be a function of x^4 , and x^4 must be a function of that co-ordinate only.

Now apply condition (13). Putting $\mu = \nu$ in (14) we have

$$\bar{T}_{\mu}{}^{\mu} = T_1{}^1 + (T_4{}^4 - T_1{}^1) \frac{\partial \bar{x}^{\mu}}{\partial x^4} \frac{\partial x^4}{\partial \bar{x}^{\mu}}. \tag{17}$$

Hence, putting $\mu = 1, 2, 3$ in turn and equating the results, we have—since $T_4^4 - T_1^1$ does not vanish—

$$\frac{\partial \bar{x}^1}{\partial x^4} \frac{\partial x^4}{\partial \bar{x}^1} = \frac{\partial \bar{x}^2}{\partial x^4} \frac{\partial x^4}{\partial \bar{x}^2} = \frac{\partial \bar{x}^3}{\partial x^4} \frac{\partial x^4}{\partial \bar{x}^3}. \quad (18)$$

If $\bar{x}^1$, $\bar{x}^2$ or $\bar{x}^3$ is the co-ordinate which must be a function of x^4 , this condition cannot be satisfied, but it is necessarily satisfied if $\bar{x}^4$ is a function of x^4 only. Hence α must be 4, and the only transformations of co-ordinates permissible under conditions (12) and (13) are those which do not associate the space-like and time-like co-ordinates with one another.

6. *Models satisfying the Assumptions Made.*—We are now in a position to consider the general problem of how the line-element (5) is specialised by the conditions (6), (7) and (8). In a previous paper,* expressions for the components of the mixed energy-momentum tensor, T_μ^ν , were given for a more general line-element. Inserting in these expressions the values of $g_{\mu\mu}$ corresponding to (5), we find on reduction that the condition that (6) shall hold is

$$2 \frac{\partial}{\partial \sigma} (L + M) - (L + M)^2 + 2M^2 = 0. \quad (19)$$

Similarly, the condition that (7) shall hold is found to be

$$\frac{\partial L}{\partial t} \equiv \frac{\partial N}{\partial \sigma} = \frac{1}{2} MN. \quad (20)$$

Condition (8) gives us the relation (19) again—indicating, as we should expect, that equality of pressure in all directions implies the vanishing of tangential stresses.

A first integral of (20) is easily obtained, namely,

$$H = [a(t)]^2 N^2. \quad (21)$$

This is effectively an equation connecting F and H , so that the physical meaning of condition (7)—namely, that axes can be chosen which move with the matter at every point—is that the relation between co-ordinate- and proper-distance shall be definitely connected with the relation between co-ordinate- and proper-time in the manner indicated by (21).

To solve our problem completely we must determine F and H from equations (19) and (21), but before attempting this it will be convenient to give the general expressions for the density and pressure of energy in the universe according to our assumptions. Evaluating T_1^1 and T_4^4 from the general expressions above referred to, and using the relation (19), we obtain

$$\left. \begin{aligned} 8\pi p &= -8\pi T_1^1 = \frac{1}{F}(2 + \sigma L)(L + M) + \frac{\sigma LM}{F} - \frac{N}{2H} \left(\frac{3}{2}N - P \right) - \frac{1}{H} \frac{\partial N}{\partial t} + \Lambda; \\ 8\pi p &= 8\pi T_4^4 = -\frac{L}{F}(6 + \sigma L) - \frac{4\sigma}{F} \frac{\partial L}{\partial \sigma} + \frac{3N^2}{4H} - \Lambda, \end{aligned} \right\} \quad (22)$$

* *Proc. Nat. Acad. Sci.*, 19, 559, 1933.

p being the co-ordinate pressure and ρ the density reckoned with respect to co-ordinates which are at rest in the matter as a whole.

Eliminating H by means of (21), these expressions reduce to the slightly simpler forms,

$$\left. \begin{aligned} 8\pi p &= \frac{1}{F}(2 + \sigma L)(L + M) + \frac{\sigma LM}{F} - \frac{1}{a^2} \left(\frac{3}{4} - \frac{1}{Na} \frac{da}{dt} \right) + \Lambda; \\ 8\pi \rho &= -\frac{L}{F}(6 + \sigma L) - \frac{4\sigma}{F} \frac{\partial L}{\partial \sigma} + \frac{3}{4a^2} - \Lambda. \end{aligned} \right\} \quad (23)$$

It may be useful to note that the mathematical condition for isotropy, (19), has been used, but not exhausted, in the expressions for p , and has not been used at all in the expressions for ρ . On the other hand, the condition (20) or (21), that the matter is stationary in the co-ordinate system, has been used for both p and ρ in (23), but for neither in (22).

I have not been able to obtain a general solution of (19) in finite terms, but in a number of special cases a finite solution is possible. For the purposes of this paper we shall require three such solutions, all of which are obtainable in sufficient generality. They correspond to the cases of small M (which, as we shall show, means very slight inhomogeneity), of homogeneity and of staticity, respectively. We proceed to derive these solutions.

7. *Solution of the Equations when M is Small.*—Equation (19), which resists general solution, becomes tractable when M is so small that its square may be neglected. In that case we have

$$2 \frac{\partial}{\partial \sigma} (L + M) = (L + M)^2, \quad (24)$$

whence

$$L + M \equiv \frac{1}{F} \frac{\partial F}{\partial \sigma} + \frac{1}{H} \frac{\partial H}{\partial \sigma} = -\frac{2}{\sigma + b(t)}. \quad (25)$$

Integrating, we have

$$\log \frac{FH}{[h(t)]^2} = -2 \log (\sigma + b), \quad (26)$$

i.e.

$$FH = \frac{h^2}{(\sigma + b)^2}. \quad (27)$$

Equation (21) provides another relation between F and H , namely,

$$H = \frac{a^2 \left(\frac{\partial F}{\partial t} \right)^2}{F^2}. \quad (28)$$

Substituting into (27) we have, therefore,

$$\frac{a}{F^{1/2}} \frac{\partial F}{\partial t} = \frac{h}{\sigma + b}; \quad (29)$$

whence

$$2F^{1/2} = \int \frac{h}{a(\sigma + b)} dt + 2\beta(\sigma), \quad (30)$$

i.e.

$$F = \left[\beta + \int \frac{h}{2a(b+\sigma)} dt \right]^2, \quad (31)$$

and, from (27),

$$H = \frac{1}{F} \frac{h^2}{(b+\sigma)^2}. \quad (32)$$

These expressions may be simplified by the transformation of the time-like co-ordinate according to the equation $hdt = d\bar{t}$, so that h disappears. We retain the quantity h as a constant, Ω , however, in order to facilitate the later representation of homogeneous and static universes, with their customary line-elements, as special cases of the present solution.

The general line-element for a not very inhomogeneous universe satisfying our conditions is therefore

$$ds^2 = - \left(\beta + \int \frac{\Omega}{2a(b+\sigma)} dt \right)^2 [dx^2 + dy^2 + dz^2] + \frac{1}{\left(\beta + \int \frac{\Omega}{2a(b+\sigma)} dt \right)^2} \frac{\Omega^2}{(b+\sigma)^2} dt^2. \quad (33)$$

For convenience of reference we shall call the universe represented by (33) the "nearly homogeneous" universe; it includes the completely homogeneous universe, of course, as a special case.

8. *The General Homogeneous Universe.*—In order to obtain the general form of line-element of a homogeneous universe, we must first express our physical condition for homogeneity, (2), in terms of the functions associated with the metric. We do this as follows. Differentiating the first equation of (23) with respect to σ , we have, after reduction,

$$8\pi \frac{\partial p}{\partial \sigma} = M \left[\frac{1}{F} (2L - M) + \frac{\sigma}{F} \left(2 \frac{\partial L}{\partial \sigma} - LM \right) - \frac{1}{2Na^3} \frac{da}{dt} \right]. \quad (34)$$

Adding equations (23), we obtain

$$8\pi(p+\rho) = -\frac{4L}{F} + \frac{2M}{F} - \frac{4\sigma}{F} \frac{\partial L}{\partial \sigma} + \frac{2\sigma LM}{F} + \frac{1}{Na^3} \frac{da}{dt}; \quad (35)$$

i.e.

$$-4\pi(p+\rho) = \frac{1}{F} (2L - M) + \frac{\sigma}{F} \left(2 \frac{\partial L}{\partial \sigma} - LM \right) - \frac{1}{2Na^3} \frac{da}{dt}. \quad (36)$$

Hence, substituting in (34), we obtain

$$\frac{\partial p}{\partial \sigma} = -\frac{M}{2}(p+\rho). \quad (37)$$

Now $(p+\rho)$ cannot be negative, for physical reasons. Hence, except in the case of an empty universe, the necessary and sufficient condition that $\frac{\partial p}{\partial \sigma} = 0$ is that $M = 0$; *i.e.* that the component g_{44} of the fundamental tensor, which determines the relation between co-ordinate- and proper-time, is independent of the distance. Further, we see that when this condition does not hold, the sign of $\frac{\partial p}{\partial \sigma}$ is definitely determined by that of M .

We shall now show that the condition $M=0$ requires that $\frac{\partial \rho}{\partial \sigma}=0$ also. Differentiating the second equation of (23) with respect to σ , we have

$$8\pi \frac{\partial \rho}{\partial \sigma} = -\frac{1}{F} \left[4\sigma \frac{\partial^2 L}{\partial \sigma^2} + 2(5 - \sigma L) \frac{\partial L}{\partial \sigma} - L^2(5 + \sigma L) \right]. \quad (38)$$

Now when $M=0$, (19) becomes

$$2 \frac{\partial L}{\partial \sigma} - L^2 = 0; \quad (39)$$

i.e.

$$2 \frac{\partial^2 L}{\partial \sigma^2} = 2L \frac{\partial L}{\partial \sigma} = L^3. \quad (40)$$

Hence (38) becomes

$$8\pi \frac{\partial \rho}{\partial \sigma} = -\frac{1}{F} [2\sigma L^3 + L^2(5 - \sigma L) - L^2(5 + \sigma L)] \equiv 0. \quad (41)$$

Hence, when M vanishes, $\frac{\partial \rho}{\partial \sigma}$ and $\frac{\partial \rho}{\partial \sigma}$ also vanish. Now if M is not zero in our system of co-ordinates, it is clear that no transformation of the kind allowed by paragraph 5 can make it vanish. It follows that the necessary and sufficient condition that a universe satisfying the conditions assumed shall satisfy the further condition of spatial homogeneity is that

$$M \equiv \frac{\partial}{\partial \sigma} \log H = 0. \quad (42)$$

The one exception to this rule is the de Sitter universe, in which $(p + \rho) = 0$. Here M need not vanish for homogeneity, but we shall refer to this again later.

When $M=0$ equations (19) and (20) are easily soluble. From (20) we have

$$\frac{\partial L}{\partial t} = 0; \quad \text{i.e. } L = \phi(\sigma). \quad (43)$$

Substituting in (19), which now takes the form (39), we obtain

$$2 \frac{\partial \phi}{\partial \sigma} = \phi^2. \quad (44)$$

The solution of this is

$$\phi \equiv \frac{d}{d\sigma} \log F = -\frac{2}{\Omega + \sigma}, \quad (45)$$

whence

$$F = \frac{[k(t)]^2}{(\Omega + \sigma)^2}. \quad (46)$$

The value of H follows from (21), namely,

$$H = a^2 \left[\frac{\partial}{\partial t} \log F \right]^2 = \left(\frac{2a}{k} \frac{dk}{dt} \right)^2. \quad (47)$$

To express the line-element in the usual form, choose a new variable, $\bar{t}$, satisfying the relation,

$$\frac{d\bar{t}}{dt} = 2a \frac{d}{dt} \log k. \quad (48)$$

Then we have

$$H dt^2 = d\bar{t}^2, \quad (49)$$

and

$$k = e^{\int \frac{1}{2a} \frac{d\bar{t}}{dt} dt} = \bar{k}, \text{ say,} \quad (50)$$

when the variable is transformed by means of (48). With this new co-ordinate, therefore, we have

$$\left. \begin{aligned} F &= \frac{\bar{k}^2}{(\Omega + \sigma)^2}; \\ H &= 1, \end{aligned} \right\} \quad (51)$$

and the line-element for the spatially homogeneous universe becomes

$$ds^2 = - \frac{\bar{k}^2}{(\Omega + \sigma)^2} [dx^2 + dy^2 + dz^2] + dt^2, \quad (52)$$

the symbol for the new time-like co-ordinate being altered to t , to conform with custom.

This is the Friedmann-Lemaître line-element, which is often * written in the form

$$ds^2 = - \frac{e^{g(t)}}{\left(1 + \frac{\sigma}{4R^2} \right)^2} [dx^2 + dy^2 + dz^2] + dt^2. \quad (53)$$

By comparison we see that

$$\left. \begin{aligned} \Omega &= 4R^2; \\ \bar{k}^2 &= 16R^4 e^g. \end{aligned} \right\} \quad (54)$$

It is to be noted, however, that in the general form Ω (or $4R^2$) may be real, imaginary, positive, negative, or infinite. It is usually taken to be real, finite and positive, but, as de Sitter particularly has emphasised, there is no logical necessity for this.

The present investigation shows that the line-element (52) includes all possible spatially homogeneous universes satisfying the assumptions of paragraph 3. It is probably the physical equivalent of the mathematical investigation of Robertson,† who derived the same line-element as the most general one which satisfied certain abstract conditions.

* E.g. Tolman, *Proc. Nat. Acad. Sci.*, **16**, 582, 1930, and elsewhere.

† *Proc. Nat. Acad. Sci.*, **15**, 822, 1929.

9. *The General Static Universe.*—The condition for staticity is that the pressure and density shall be independent of the time. Differentiating the second equation of (23) with respect to t , we have, after reduction,

$$8\pi \frac{\partial \rho}{\partial t} = 3N \left[\frac{1}{F} (2L - M) + \frac{\sigma}{F} \left(2 \frac{\partial L}{\partial \sigma} - LM \right) - \frac{1}{2Na^3} \frac{da}{dt} \right], \quad (55)$$

whence, from (36),

$$\frac{\partial \rho}{\partial t} = -\frac{3N}{2} (p + \rho). \quad (56)$$

This equation is of precisely the same form as (37), and the correspondence clearly brings out the fact that the relation between pressure and space is analogous to that between density and time. The factor 3 which differentiates (56) from (37) obviously appears because space has three dimensions and time one.

It follows, therefore, that, except in the case of an empty universe, the necessary and sufficient condition that the density shall be independent of the time (and therefore a necessary condition that space shall be static) is that $N=0$, *i.e.* that the function F , which determines the relation between co-ordinate- and proper-distance, is independent of the time. When this condition does not hold, the sign of $\frac{\partial \rho}{\partial t}$ is definitely determined by that of N .

We must now see if this is a sufficient condition for staticity. Differentiating the first equation of (23) with respect to t and putting $N=0$ in the result, we obtain

$$8\pi \frac{\partial p}{\partial t} = \frac{2}{F} \frac{\partial M}{\partial t} (1 + \sigma L). \quad (57)$$

(In deriving this we use equation (20), and note that when $N=0$, (21) requires that $a=\infty$.) Hence, in order that the pressure as well as the density shall be constant we must have either

$$(i) \quad \frac{\partial M}{\partial t} = 0 \quad (58)$$

or

$$(ii) \quad 1 + \sigma L = 0. \quad (59)$$

Now the second case can easily be shown to correspond to the empty de Sitter universe, for we have

$$L \equiv \frac{\partial}{\partial \sigma} \log F = -\frac{1}{\sigma}, \quad (60)$$

whence

$$F = \frac{\Gamma}{\sigma}, \quad (61)$$

Γ being a constant independent of t since N is zero. Substituting these values in (23) we obtain at once

$$\left. \begin{aligned} 8\pi p &= -\frac{1}{\Gamma} + \Lambda; \\ 8\pi p &= \frac{1}{\Gamma} - \Lambda, \end{aligned} \right\} \quad (62)$$

which are the well-known zero values for the de Sitter universe. Consequently, if a world not devoid of matter and energy is to be static we must have

$$\left. \begin{aligned} N &= 0, \\ \frac{\partial M}{\partial t} &= 0. \end{aligned} \right\} \quad (63)$$

The geometrical meaning of $N=0$ is that F is a function of σ only. The meaning of $\frac{\partial M}{\partial t}=0$ is that H , in its most general form, shall be a product of functions of σ only and t only. Now the function of t in such a case can always be transformed into unity by a change of co-ordinates permitted by paragraph 5, so that the geometrical meaning of (63) is that a co-ordinate system is possible in which both F and H are independent of t , *i.e.* in which the relations between co-ordinate- and proper-distances and times are both independent of the time. Equations (63), therefore, constitute the necessary and sufficient conditions for staticity.

The most general line-element for a static universe is now obvious. It is

$$ds^2 = -[\theta(\sigma)]^2[dx^2 + dy^2 + dz^2] + [\gamma(\sigma)q(t)]^2 dt^2, \quad (64)$$

where, as already mentioned, q can be eliminated by the transformation, $qdt = d\bar{t}$, and θ and γ are related by equation (19).

10. *Homogeneous and Static Universes as Special Forms of the Nearly Homogeneous Universe.*—The line-element (33) clearly includes homogeneous and static universes as special cases—provided the latter, if not completely homogeneous, depart only slightly from the homogeneous state. It will be useful to see how the various functions occurring in (33) become specialised in these particular cases. Taking the homogeneous universe first, we have, from (25), on putting $M=0$,

$$L \equiv \frac{1}{F} \frac{\partial F}{\partial \sigma} = -\frac{2}{b + \sigma}, \quad (65)$$

whence, on integration,

$$F = \frac{[f(t)]^2}{(b + \sigma)^2}. \quad (66)$$

Equating this to the general value of F in (33) we have

$$\frac{f}{b + \sigma} = \beta + \int \frac{\Omega}{2a(b + \sigma)} dt. \quad (67)$$

This can be satisfied if $\beta=0$ and b is constant, in which case f becomes $\Omega \int \frac{dt}{2a}$. Such a solution, however, as we shall see, does not provide a

continuous passage from the general homogeneous universe to the static homogeneous (Einstein) universe, and we therefore choose the alternative identifications,

$$b = \Omega ; \quad (68)$$

$$\beta = \frac{\Omega}{\Omega + \sigma}, \quad (69)$$

whence

$$f = \Omega \left(1 + \int \frac{dt}{2a} \right). \quad (70)$$

(The choice of Ω for the constant value of b involves no loss of generality, since Ω merely determines the unit of measurement.)

The line-element of a homogeneous universe therefore takes the form

$$ds^2 = - \left[\frac{\Omega}{\Omega + \sigma} \left(1 + \int \frac{dt}{2a} \right) \right]^2 (dx^2 + dy^2 + dz^2) + \frac{1}{\left(1 + \int \frac{dt}{2a} \right)^2} dt^2. \quad (71)$$

When departure from homogeneity is not under consideration, H can always be made unity by an obvious transformation.

For the static universe we put $N = 0$ and $\frac{\partial M}{\partial t} = 0$. The first condition, on account of (21), gives us

$$a = \infty, \quad (72)$$

whence, from (33),

$$F = \beta^2 ; \quad (73)$$

$$H = \frac{\Omega^2}{\beta^2(b + \sigma)^2}. \quad (74)$$

From the second condition, and equations (25) and (20), we have

$$\frac{\partial M}{\partial t} = - \frac{\partial L}{\partial t} - \frac{\partial}{\partial t} \left(\frac{2}{b + \sigma} \right) = - \frac{1}{2} MN + \frac{2}{(b + \sigma)^2} \frac{db}{dt} = 0, \quad (75)$$

and since $N = 0$ this becomes

$$\frac{2}{(b + \sigma)^2} \frac{db}{dt} = 0. \quad (76)$$

Now b cannot be infinite, or, from (74), H would vanish.* Hence we must have $\frac{db}{dt} = 0$; i.e. b must be constant. We choose, therefore, as before,

$$b = \Omega. \quad (77)$$

Since β is still arbitrary, we choose a new arbitrary function $\mu(\sigma)$ such that

$$\beta \equiv \frac{\Omega}{\Omega + \sigma} \mu. \quad (78)$$

* This is not necessarily true if $\Omega = \infty$, but that would restrict us to a special type of static universe, namely, one in which space is infinite, and we are now considering the most general case.

Equations (73) and (74) then become

$$F = \left(\frac{\Omega\mu}{\Omega + \sigma} \right)^2; \quad (79)$$

$$H = \frac{1}{\mu^2}, \quad (80)$$

and the line-element of the general static and not very inhomogeneous universe takes the form

$$ds^2 = - \left(\frac{\Omega\mu}{\Omega + \sigma} \right)^2 (dx^2 + dy^2 + dz^2) + \frac{1}{\mu^2} dt^2. \quad (81)$$

For a universe (the Einstein universe) which is both homogeneous and static, equations (71) and (81) become identical when we put

$$1 + \int \frac{dt}{2a} = \mu = 1. \quad (82)$$

It will be useful to collect these results for future reference. This is done in Table I.

TABLE I

Function (i)	General Nearly Homogeneous Universe (ii)	Homogeneous Universe (Friedmann- Lemaître) (iii)	Static and Nearly Homogeneous Universe (iv)	Homogeneous and Static Universe (Einstein) (v)
F	$\left[\beta + \int \frac{\Omega}{2a(b+\sigma)} dt \right]^2$	$\left[\frac{\Omega}{\Omega + \sigma} \left(1 + \int \frac{dt}{2a} \right) \right]^2$	$\left[\frac{\Omega\mu}{\Omega + \sigma} \right]^2$	$\left[\frac{\Omega}{\Omega + \sigma} \right]^2$
H	$\frac{\Omega^2}{F(b+\sigma)^2}$	$\frac{1}{\left(1 + \int \frac{dt}{2a} \right)^2}$	$\frac{1}{\mu^2}$	1
L	$2 \left[\frac{d\beta}{d\sigma} + \frac{\partial}{\partial \sigma} \int \frac{\Omega}{2a(b+\sigma)} dt \right]$ $\beta + \int \frac{\Omega}{2a(b+\sigma)} dt$	$-\frac{2}{\Omega + \sigma}$	$2 \left[\frac{1}{\mu} \frac{d\mu}{d\sigma} - \frac{1}{\Omega + \sigma} \right]$	$-\frac{2}{\Omega + \sigma}$
M	$-L - 2 \frac{\partial}{\partial \sigma} [\log(b+\sigma)]$	0	$-\frac{2}{\mu} \frac{d\mu}{d\sigma}$	0
N	$\frac{H^{1/2}}{a}$	$\frac{1}{a \left(1 + \int \frac{dt}{2a} \right)}$	0	0
P	$-N - 2 \frac{\partial}{\partial t} [\log(b+\sigma)]$	$-\frac{1}{a \left(1 + \int \frac{dt}{2a} \right)}$	0	0
a	a	a	$\pm \infty$	$\pm \infty$
b	b	Ω	Ω	Ω
β	β	$\frac{\Omega}{\Omega + \sigma}$	$\frac{\Omega}{\Omega + \sigma^\mu}$	$\frac{\Omega}{\Omega + \sigma}$

11. *Interpretation of the Arbitrary Functions—Cosmic Time.*—The general line-element (33) contains three arbitrary functions— a , β , b —which Table I assists us in interpreting.

From the relation, $H^2 = aN$, it is clear that a is the factor which indicates departure of the universe from the static state: it is the quantity by which $H^{1/2}$ must be divided in order to give N , the measure of the rate of change of density. In the static state $a = \infty$: in a non-static universe, positive a indicates expansion and negative a contraction.

β , on the other hand, is a criterion of the departure of the universe from the homogeneous state. In any homogeneous universe, whether static or not, it is equal to $\frac{\Omega}{\Omega + \sigma}$. In a static but non-homogeneous universe this

value is multiplied by μ , a factor which measures M , the degree of inhomogeneity. In a universe which is neither static nor homogeneous it is in general any arbitrary function of σ , and may therefore still be represented

by $\frac{\Omega}{\Omega + \sigma}\mu$, where μ differentiates one such universe from another. The functions a and β therefore—one purely temporal and the other purely spatial—may be regarded as indices of distinction between the various types of non-static and inhomogeneous universes respectively.

The function b is not so simple. In an Einstein universe it is constant and equal to Ω , *i.e.* to $4R^2$, where R is the radius. In a homogeneous but non-static universe it is still constant, and may be regarded as the initial radius of such a universe. In the general form it is no longer necessarily constant, but we may obtain some indication of its meaning by considering what its constancy would indicate. If we equate it to Ω , then from (25) we have

$$\left. \begin{aligned} \frac{\partial L}{\partial t} &= -\frac{\partial M}{\partial t} \\ &= \frac{1}{2} MN, \text{ from (20),} \end{aligned} \right\} \quad (83)$$

whence, by direct integration,

$$F = \frac{1}{[\xi(\sigma)]^2 M^2}. \quad (84)$$

Now this is analogous to (21) with the spatial and temporal co-ordinates interchanged, just as (83) is similarly analogous to (20). Since, therefore, (20) expresses the condition that the co-ordinate system moves with the matter—*i.e.* that as we progress in time the matter of the universe keeps the same co-ordinate position—so we may regard (83) or (84) as expressing the condition that as we progress in space the matter of the universe keeps the same co-ordinate time. In other words, (84) is the condition that the universe admits of a time system consisting of world-wide instants—the so-called “cosmic time” which has aroused so much discussion.

It is to be noticed, however, that this “cosmic time” involves no violation of the principle of relativity. It is a possibility only, not a necessity. We are no more compelled to adopt such a time system than to choose axes moving with the matter; the two things, in fact, are on precisely the same

footing. In a homogeneous or static universe, as Table I shows, the conditions assumed in § 3 are sufficient to ensure a cosmic time system without further assumption, for b is constant in each of these cases. The general nearly homogeneous universe, however, has not this characteristic, and if we wish on physical grounds to restrict our considerations to those examples of such universes which do possess it, we must impose this condition as an additional assumption. In this paper we shall do so, justifying our action by the consideration that a universe in which simultaneity had no possible significance would be too remote from ordinary experience to be counted possible in the present state of our knowledge.

We can now rewrite Table I in a simpler form, showing more clearly the relation of the special cases of stability and homogeneity to the general nearly homogeneous universe. Putting

$$\left. \begin{aligned} b &= \Omega, \\ v &= \int \frac{dt}{2a}, \\ \beta &= \frac{\Omega}{\Omega + \sigma} \mu, \end{aligned} \right\} \quad (85)$$

and

where v and μ are arbitrary functions of t and σ , respectively, the table takes the following form:—

TABLE II

Function (i)	General Nearly Homogeneous Universe (ii)	Homogeneous Universe (Friedmann- Lemaître) (iii)	Static and Nearly Homogeneous Universe (iv)	Homogeneous and Static Universe (Einstein) (v)
F	$\left[\frac{\Omega}{\Omega + \sigma} (\mu + v) \right]^2$	$\left[\frac{\Omega}{\Omega + \sigma} (1 + v) \right]^2$	$\left[\frac{\Omega}{\Omega + \sigma} \mu \right]^2$	$\left[\frac{\Omega}{\Omega + \sigma} \right]^2$
H	$\frac{1}{(\mu + v)^2}$	$\frac{1}{(1 + v)^2}$	$\frac{1}{\mu^2}$	1
L	$\frac{2}{\mu + v} \frac{d\mu}{d\sigma} - \frac{2}{\Omega + \sigma}$	$-\frac{2}{\Omega + \sigma}$	$\frac{2}{\mu} \frac{d\mu}{d\sigma} - \frac{2}{\Omega + \sigma}$	$-\frac{2}{\Omega + \sigma}$
M	$-\frac{2}{\mu + v} \frac{d\mu}{d\sigma}$	0	$-\frac{2}{\mu} \frac{d\mu}{d\sigma}$	0
N	$\frac{2}{\mu + v} \frac{dv}{dt}$	$\frac{2}{1 + v} \frac{dv}{dt}$	0	0
P	$-\frac{2}{\mu + v} \frac{dv}{dt}$	$-\frac{2}{1 + v} \frac{dv}{dt}$	0	0
a	$\frac{1}{2} \frac{dv}{dt}$	$\frac{1}{2} \frac{dv}{dt}$	$\pm \infty$	$\pm \infty$
b	Ω	Ω	Ω	Ω
β	$\frac{\Omega}{\Omega + \sigma} \mu$	$\frac{\Omega}{\Omega + \sigma}$	$\frac{\Omega}{\Omega + \sigma} \mu$	$\frac{\Omega}{\Omega + \sigma}$

The general line-element may now be written

$$ds^2 = - \left[\frac{\Omega}{\Omega + \sigma} (\mu + v) \right]^2 (dx^2 + dy^2 + dz^2) + \frac{1}{(\mu + v)^2} dt^2, \quad (86)$$

the condition for homogeneity being simply

$$\mu = 1, \quad (87)$$

and the condition for staticity,

$$v = 0. \quad (88)$$

12. *Stability of a Static Universe.*—Suppose a slight disturbance occurs in a static universe which is at most not very inhomogeneous. It passes over at once into the general case represented by (86), and its stability depends on whether it automatically returns to its former state or not. We shall limit ourselves here to the case of a universe which is originally homogeneous as well as static, since the severe restriction which our special solution places on the degree of inhomogeneity for which (33) holds robs the slightly more general case of most of its additional interest.

The stability of the static state of such a universe, under the condition that homogeneity is necessarily to be preserved, has already been investigated by Eddington and others,* and, as is well known, staticity in these circumstances is unstable. It is questionable, however, if this result is applicable to the past history of the actual universe, for the condition of permanent homogeneity is one which seems practically impossible of fulfilment. Problems concerning the origin of the universe are only doubtfully open to scientific treatment at present, but if in such speculative realms we may speak figuratively of the most likely kind of disturbance to which an Einstein universe would be susceptible, we must acknowledge it to be the establishment of a slight excess or defect of matter at some particular place, arising from what is usually called a chance configuration of the elementary particles. It must be remembered that in dealing with the universe we are not at liberty to make the assumption—perfectly valid in discussions of less complete isolated mechanical systems—that a slight disturbance of equilibrium may be caused by some unspecified external agency. There is no natural agency outside the universe, and if the homogeneous static state were ever an actuality it could have been disturbed—so far as our present physical ideas permit us to judge—only by some such chance occurrence or by supernatural means. This was fully recognised by Eddington, who sought to preserve naturalism by postulating the condensation of the original uniform distribution of matter into galaxies as the disturbance which set the universe expanding. But that such a condensation would take place simultaneously at an indefinitely large number of points, distributed throughout the universe in such a way as to keep the density and pressure on the large scale perfectly homogeneous, seems so wildly improbable as to afford a merely nominal escape from super-

* Eddington, *M.N.*, 90, 668, 1930. See also Robertson's excellent report on "Relativistic Cosmology," *Rev. Mod. Phys.*, 5, 62, 1933.

naturalism; while the subsequent attempts of McCrea and McVittie* to follow out the consequences of such an occurrence, involving as they do the immaculate conception of fully developed mass-points by a universe left with no trace of the process of childbirth, deny even lip-service to the natural order of things. It seems evident that if we are to postulate an Einstein universe as a reality of our history, we must look for its destruction to a single condensation or rarefaction large enough not to be negligible on the scale according to which the original universe was homogeneous.† But this involves the destruction, at the same time, of the homogeneity and the staticity of the original state, and the problem presents itself whether the universe, by its natural reaction, will regain either, neither or both of these qualities. We will investigate the stability of the static state first.

From our previous results it follows that the rate of change of density inherent in the process we are considering is measured by the value of N , and the growth or decline of the change depends on whether the value of $\frac{\partial N}{\partial t}$ in the nearly homogeneous universe which immediately comes into being has the same or the opposite sign to N . We therefore investigate the sign of $\frac{\partial N}{\partial t}$. From equations (22), with $b = \Omega$, we have

$$\begin{aligned} 8\pi(p + \rho) = & -\frac{4}{F}(L + M) + \frac{6M}{F} - \frac{2\sigma}{F}\left(2\frac{\partial L}{\partial \sigma} - LM\right) + \frac{1}{H}\left(\frac{1}{2}PN - \frac{\partial N}{\partial t}\right) \\ = & \frac{8}{F(\Omega + \sigma)} + \frac{6M}{F} - \frac{2\sigma}{F}\left(2\frac{\partial L}{\partial \sigma} - LM\right) + \frac{1}{H}\left(\frac{1}{2}PN - \frac{\partial N}{\partial t}\right). \end{aligned} \quad (89)$$

The disturbance will presumably affect the density and pressure of the matter throughout the whole universe, but to avoid making any assumption concerning its general effect we will confine our attention to the space origin where the conditions are definitely determined by our postulate. Putting $\sigma = 0$, therefore, the last equation becomes

$$8\pi(p + \rho) = \frac{8}{F\Omega} + \frac{6M}{F} + \frac{1}{H}\left(\frac{1}{2}PN - \frac{\partial N}{\partial t}\right), \quad (90)$$

where the various quantities involved are to be evaluated at the origin.

We consider the beginning of the disturbance, in which the density and pressure have not appreciably changed from the original values, though the rate of change of density, measured by N , is finite. The value of $8\pi(p + \rho)$ is then $\frac{8}{\Omega}$; H and F (at the origin) are unity, M is zero and P is $-N$.

* *M.N.*, 91, 128, 1930. See also subsequent papers in *M.N.* by McVittie alone.

† There is, of course, the further possibility, which has been treated by Tolman, that the cause of disturbance was a slight conversion of matter into radiation. If we knew more of the circumstances in which such a process could occur (if it can occur at all), we might find that this was much more probable than the event here suggested, but in our present ignorance it seems best to follow a more familiar line of ideas.

Equation (90) therefore becomes

$$\frac{8}{\Omega} = \frac{8}{\Omega} - \frac{1}{2}N^2 - \frac{\partial N}{\partial t}, \quad (91)$$

whence

$$\frac{\partial N}{\partial t} = -\frac{1}{2}N^2. \quad (92)$$

$\frac{\partial N}{\partial t}$ is therefore necessarily negative, whatever the sign of N . It follows that a disturbance in which N is negative (*i.e.* a condensation) will grow, while one in which N is positive (a rarefaction) will not. The reaction of the universe to the latter type of disturbance, however, would presumably carry it past the equilibrium point into a condensation, and this would develop, so that the static state of the Einstein universe is unstable for both types of disturbance.

We cannot, however, apparently obtain an expanding universe from such a disturbance. Both condensations and rarefactions lead ultimately to a negative value of N —*i.e.* a positive value of $\frac{\partial \rho}{\partial t}$ —at the origin, and the density cannot continue to grow at the origin without doing so elsewhere also. An isolated disturbance, therefore, tends to bring about a collapse of the Einstein universe about the disturbed point. We cannot pursue the course of events beyond the initial tendencies because our equations are valid only when M is very small, but it seems that if the present universe were ever in the Einstein state, it could not have expanded from it by reason of a condensation or rarefaction of matter.

13. *Stability of the Homogeneous State.*—The stability or otherwise of homogeneity in a universe satisfying our conditions follows at once from equation (83), which gives

$$\frac{\partial M}{\partial t} = -\frac{1}{2}MN. \quad (93)$$

If N is negative (*i.e.* if the universe is contracting) $\frac{\partial M}{\partial t}$ has the same sign as M , so that if a slight inhomogeneity arises in such a universe, it will develop and the homogeneous state will not be recovered. In an expanding universe, on the other hand, N is positive, and a departure from homogeneity is at once opposed. If, then, the present universe is expanding and satisfies the conditions imposed in this paper, it must be spatially homogeneous if it ever were in that condition. This conclusion does not necessarily hold if the universe were ever more than slightly inhomogeneous, for the reason already mentioned—that our results are inapplicable when M^2 is not negligible compared with M .

14. *Spectrum Displacements in a Static Universe.*—The universe of experience is generally assumed to be expanding because of the red shift of the spectrum lines of distant nebulae. This phenomenon, however,

does not in itself distinguish an expanding universe from a static inhomogeneous universe, for in the latter also spectrum lines from a distant source will necessarily be displaced with respect to local spectrum lines from the same type of atom. For the periods of vibration of similar atoms on the Earth and at a distance from the Earth will be in the inverse ratio of the square roots of the values of H at the two places, and since H is a function of σ when the universe is inhomogeneous, this ratio cannot be unity. Furthermore, since the universe is static, this effect cannot be compensated for by a variable velocity of light. Now if the universe, originally in the Einstein state, were disturbed, and thereby, as we have seen, caused to contract, it would presumably come ultimately to an inhomogeneous static state; for contraction, unlike expansion, can obviously not continue indefinitely at a finite rate. It is important to inquire, therefore, whether the well-known law of nebular recession, according to which the spectrum displacement towards the red is proportional to the distance of the source, can be a characteristic of an inhomogeneous static universe.

From equation (64), when q is transformed away, we have $H = \gamma^2$, so that, using the suffixes 0 and 1 to denote the Earth and a nebula at distance r , respectively, and representing wave-lengths by λ , we have

$$\frac{\lambda_1}{\lambda_0} = \frac{\gamma_1}{\gamma_0}, \quad (94)$$

whence

$$\Delta\lambda \equiv \lambda_1 - \lambda_0 = \left(\frac{\gamma_1}{\gamma_0} - 1 \right) \lambda_0. \quad (95)$$

If this is to be proportional to the distance r , we must have

$$\frac{\gamma_1}{\gamma_0} - 1 = \Psi r, \quad (96)$$

where Ψ is a constant. Now if γ_0 is not unity we can always transform the space co-ordinate so that it becomes so. We have then (retaining r as the symbol for the transformed co-ordinate)—

$$\gamma_1 = 1 + \Psi r = 1 + \Psi \sqrt{\sigma}; \quad (97)$$

whence

$$M \equiv \frac{1}{\gamma^2} \frac{d\gamma^2}{d\sigma} = \frac{2}{1 + \Psi \sqrt{\sigma}} \frac{\Psi}{2\sqrt{\sigma}} = \frac{\Psi}{\sqrt{\sigma}(1 + \Psi \sqrt{\sigma})}, \quad (98)$$

and

$$L = \frac{2}{\theta} \frac{d\theta}{d\sigma}. \quad (99)$$

Hence, from (23),

$$8\pi p = \frac{2}{\theta^2} \left[\frac{2}{\theta} \frac{d\theta}{d\sigma} + \frac{\Psi}{\sqrt{\sigma}(1 + \Psi \sqrt{\sigma})} \right] + \frac{2\sigma}{\theta^3} \frac{d\theta}{d\sigma} \left[\frac{2}{\theta} \frac{d\theta}{d\sigma} + \frac{2\Psi}{\sqrt{\sigma}(1 + \Psi \sqrt{\sigma})} \right] + \Lambda \left. \vphantom{\frac{2}{\theta^2}} \right\} \quad (100)$$

$$= \frac{4}{\theta^3} \frac{d\theta}{d\sigma} \left(1 + \frac{\sigma}{\theta} \frac{d\theta}{d\sigma} \right) + \frac{2\Psi}{\theta^2 \sqrt{\sigma}(1 + \Psi \sqrt{\sigma})} \left(1 + \frac{2\sigma}{\theta} \frac{d\theta}{d\sigma} \right).$$

Hence, at the origin, where $\sigma=0$, p becomes infinite. Consequently, in a static universe having a finite pressure at the origin, it is impossible to have a spectrum displacement proportional to the distance of the source. The case is obviously not improved if we suppose (96) to be only a first approximation and add terms involving higher powers of r . We conclude, therefore, that the phenomenon observed in the spectra of the extra-galactic nebulae would not be possible in a static universe, and hence that the universe we know is not static.

It should be noticed that this result is independent of the restriction to slight inhomogeneity and to cosmic time, and applies to any static universe satisfying the conditions of § 3.

The net result of our investigation, therefore, is that if the universe satisfies the conditions of § 3 it must be expanding. If, further, it necessarily permits of a time system in which universal simultaneity has a meaning, the universe, if ever spatially homogeneous, must remain so apart from supernatural interference.

15. *Examination of the Reasons for which the Universe has hitherto been assumed to be Homogeneous.*—The foregoing discussion, while undoubtedly giving support to the idea that the universe is spatially homogeneous, by no means excludes the alternative possibility. Our investigation not only throws no light on the subsequent behaviour of a universe which began in an appreciably inhomogeneous state, but does, in fact, to some extent clear the ground for such an origin by showing that the present universe could not have developed from the static, homogeneous state by any natural process which can be accorded a reasonable degree of probability. Furthermore, apart from this possibility, equation (93) itself provides for a temporary slight inhomogeneity in that it represents an oscillation about the stable homogeneous state. It is not out of place, therefore, to review the arguments for spatial homogeneity, not so much in order to throw light on the condition of the universe now as to clarify the reasons for which the assumption of homogeneity may properly be justified. We must categorically dissent from the extreme idea held by Milne,* that homogeneity is included in the definition of the universe, and that "the only way we recognise any proposed system as suitable for representing the world is by verifying that it possesses the property first enunciated by Einstein, that all 'places' in it are equivalent." We take it to be perfectly conceivable that an increase of telescopic power may reveal a variation of material density with distance, and the denial of this possibility connoted by Milne's statement seems to us to be inconsistent with the fundamental principles of science. We hold that the assumption of spatial homogeneity is of precisely the same character as the earlier assumption of the homogeneity of the four-dimensional world—a working hypothesis, valid so long as it does not conflict with observation or with theoretical probability, and justifiable during that time as a restriction on arbitrary speculation. We shall consider, therefore, how far observation and theoretical probability now make alternative assumptions feasible.

The general spatially homogeneous universe of Friedmann and Lemaître

* See, for example, *Zeits. f. Astroph.*, 7, 185, 1933.

is in approximate agreement with observation, but there are at least two points on which its requirements raise difficulties. In the first place, the highest density of energy, including matter, which it seems able to permit is lower than that which Hubble estimates from observation of the visible nebulae; and future quite possible, and even probable, discoveries of dark nebulae and inter-galactic dust, gas or radiation, would only increase the discrepancy. Secondly, the time-scale implied by the metric in question, as is well known, is at variance with that strongly indicated by several astronomical phenomena.* While it would be premature to conclude that the Friedmann-Lemaître universe is definitely in conflict with observation, these difficulties are nevertheless sufficiently great to warrant an attempt to remove them by dropping the assumption of homogeneity. If a non-homogeneous solution of the field equations can be obtained which explains the red shift of the nebular spectrum lines as well as does the solution of Friedmann and Lemaître, and at the same time has more satisfactory requirements with regard to these two points, it will have at least equal claims to consideration as the best-known approximation to the actual universe.

There is a philosophical aspect of the matter, however, which demands attention. It is sometimes asserted that since, in a non-homogeneous space, one point can be distinguished from another, space may be said to have a physical existence, which is contrary to the principle of relativity. The principle of relativity, however, does not require that two points of space shall be indistinguishable, but only that they shall be indistinguishable without reference to matter: they shall have no metaphysical distinguishing marks such as would be provided by the older conception of an all-pervading æther. Relativity states that space is relative to the distribution of matter in the universe; it makes no declaration of what that distribution shall be. If space is not homogeneous, therefore, to say that one place can be distinguished from another is merely to state the principle of relativity, not to violate it. Furthermore, whatever philosophical objections may be advanced against the inhomogeneity of space tell with equal force against the inhomogeneity of time, and that time is not homogeneous (*i.e.* that space is not static) is already acknowledged in the acceptance of the Friedmann-Lemaître model of the universe. Indeed, if complete reciprocity of space and time be regarded (as it might well be) as a necessary part of the philosophical principle of relativity, the only possible homogeneous space is a static one, and the admission of the expansion of space commits us also to the admission of its inhomogeneity. We are not concerned here, however, with philosophical arguments for a particular conclusion, but only to meet possible philosophical objections to an extended freedom of inquiry.

Undoubtedly it is curious, if the universe is not homogeneous, that we should be apparently at a point from which the nebulae recede uniformly

* For a summary of the evidence on this matter, see *P.A.S.P.*, 45, 159, 1933. A subsequent paper by de Sitter (*M.N.*, 93, 628, 1933) suggests the further possibility that the stars are older than the universe, but if that were so, the epoch 2×10^9 years ago, when the nebulae appear to have begun their recession, would seem to be at least as difficult to interpret as the conflicting time-scales themselves.

in all directions. It must be noted, however, that the evidence for this is far from complete, owing particularly to lack of observations in the southern skies. Also, even if the peculiarity should be established by more thorough investigations, it by no means follows that we are in a unique position in the universe as a whole. It will be remembered that star-counts show that the density of stellar distribution in our neighbourhood decreases with distance in all directions from the Earth, but we do not now regard this as evidence that we are at the centre of the galactic system, although the counts are undisputed. Again, even if the solar system is actually at the centre of dispersion of the nebulae, and the dispersion itself should turn out to be a universal phenomenon, the assumption of a non-homogeneous universe would give us no more improbable a position in space than the Friedmann-Lemaître universe gives us in time. According to that model and the evidence of geological observation, the formation of the Earth occurred approximately at a unique point in the history of the universe, namely, its beginning. According to an inhomogeneous model and the evidence of nebular observation, the position of the Earth would be approximately at a unique point in the structure of the universe, namely, its centre. The two situations are precisely parallel, and it is difficult to see how a non-spatially homogeneous model can be rejected on these grounds if a non-static model is retained.

Finally, it should be remarked that all objections, based on philosophical postulates regarding the character of space, to the representation of the observed universe by an inhomogeneous model, are vitiated by the fact that we have no grounds for supposing that the part of the universe which is observed is typical of the whole. For all we know to the contrary, the region we have explored may be merely a point in the complete universe (as, in fact, in regard to size it practically is), and the phenomena we embody in our models may be purely local characteristics. By requiring of a model that it shall yield the local density and pressure of matter and the observed red shift of nebular light, we automatically abnegate the right to regard it as *necessarily* representative of the whole actual universe. It may be so, of course, and if we are to talk at all scientifically about the universe we must assume that it is, but it is necessary to remember that we are making such an assumption. Space may be homogeneous on the large scale, while at the same time the observed facts of our neighbourhood may not be capable of representation by a homogeneous model.

We may conclude, therefore, that while, in the present state of knowledge, a spatially homogeneous universe has greater claims to consideration than any other, such claims are entirely subject to the evidence of observation and have no *a priori* justification. If an inhomogeneous model can be found which, while representing the red shift of nebular spectrum lines and the observed distribution of matter in our neighbourhood as accurately as the Friedmann-Lemaître model, at the same time has more satisfactory requirements in other respects (notably the time-scale), it will have stronger claims to acceptance.

16. *Remark concerning the de Sitter Universe.*—Equations (37) and (56) throw an interesting light on the de Sitter universe. Owing to the

factor, $(p + \rho)$, occurring in the expressions for both $\frac{\partial p}{\partial \sigma}$ and $\frac{\partial \rho}{\partial t}$, the physical conditions for both homogeneity and staticity can be satisfied without the vanishing of either M or N , provided $p + \rho = 0$, and this cannot occur unless p and ρ vanish individually, *i.e.* unless the universe is empty. The de Sitter universe can accordingly appear metrically as non-static or inhomogeneous or both, while in its mechanical aspect it necessarily has the unending sameness and immobility of the void. This discordance is associated with the fact that the vanishing of $T_{\mu}^{\mu} (\mu \neq 4)$, which we have taken as the condition that the matter shall be stationary in the co-ordinate system, is really the less stringent condition that the momentum shall vanish, and this is satisfied if there is no mass, without any restriction on the velocity. In that case, therefore, M and N are set free to take any finite values they will. In the original form of the de Sitter line-element, N happened to be zero, and the universe accordingly appeared to be static, while in the form adopted by Robertson,* M is zero, and it appears to be homogeneous. The actual position is clearly revealed by equations (37) and (56).

17. *Acknowledgments.*—I am much indebted to many conversations with Dr. Richard C. Tolman before and during the preparation of this paper, and to subsequent criticism from Sir Arthur Eddington and Dr. W. H. McCrea. None of these gentlemen, however, is to be held responsible for the ideas expressed.

18. *Summary.*—(i) The general metric of an isotropic universe which may be slightly inhomogeneous is calculated, and a physical interpretation is given of the arbitrary functions occurring in it.

(ii) Physical definitions of staticity and spatial homogeneity are given, and criteria for these conditions in terms of the metric are derived.

(iii) The possible changes of co-ordinates which may be made without violating certain assigned conditions are determined.

(iv) The general metrics for static and spatially homogeneous universes, respectively, are determined and expressed as special forms of the metric for the nearly homogeneous universe.

(v) It is shown that an Einstein universe, subjected to a local disturbance, will contract.

(vi) The impossibility of expressing, in terms of the metric of a static universe, a red shift of spectrum lines which is proportional to the distance of the source of light, is established.

(vii) It is shown that the homogeneity of an expanding universe is stable, and that of a contracting universe is unstable.

(viii) The reasons for which the universe has hitherto been assumed to be homogeneous are examined.

(ix) A note is added on the inhomogeneous and non-static aspects of the de Sitter universe.

Imperial College of Science and Technology,
South Kensington, S.W.7:
September 1933.

* *Phil. Mag.*, 5, 835, 1928.

DISTANCE IN AN EXPANDING UNIVERSE.

A. G. Walker, *Ph.D.*

(Communicated by W. H. McCrea)

1. *Introduction.*—The distance that an observer in space-time of general relativity assigns to a star was discussed in a previous paper.* Prof. E. A. Milne kindly pointed out to me recently that a result in this paper does not agree with his formula for the apparent brightness of a receding nebula, and I find that this is due to the fact that the assumed definition of distance does not give a true account of the astronomical method of calculating distance.

In the present paper is the corrected definition, giving the distance calculated from a measurement of the apparent brightness of the star, and it is found to depend on the motions of both the star and the observer. This distance is therefore unsuitable when we are dealing with such problems as the relation between distance and recessional velocity, for it is here required that for a given observer two stars at the same event should give the same distance.

It has been shown † that the distance calculated from measurements of apparent size satisfies this condition. We shall see that this distance (Δ) bears a simple relationship with the distance (Δ') given by apparent brightness, and it is therefore convenient and advisable to reduce all observed distances Δ' to the form Δ .

The present paper is mainly concerned with finding an expression for Δ in a quite general form of expanding universe. As soon as we know the exact expression for distance in terms of co-ordinates, we can consider a possible extension of Hubble's Law, and it will be seen that this may lead to a method for determining the cosmical constant and the sign of curvature in the generally accepted form of space-time.

2. *Distance from Apparent Brightness.*—Let P_0 be the position of the observer, B , when he receives light emitted by a star, A , at the position P_1 , the points P_0, P_1 necessarily lying on a null geodesic, C , in order that light may pass from P_1 to P_0 . When the observer measures the apparent brightness of the star, he actually measures the amount of energy falling on his photographic plate. We must therefore consider a thin pencil of null geodesics issuing from the star (assumed to be a point source) and meeting the observer's instantaneous space in the two-dimension cross-section defined by the size of his exposed plate. Then the amount of energy received per second by the observer is to be found from the rate at which energy is radiated by the star into the solid angle determined by the above pencil.

* *Quart. Journ. of Maths.*, 4, 71-80, 1933. See also I. M. H. Etherington, *Phil. Mag.*, Ser. 7, 15, 761-773, 1933.

† I. M. H. Etherington, *loc. cit.*

Let $D = 1 + \delta\lambda/\lambda$ be a measure of the Doppler effect, where λ is a wavelength. Then if N_ν is the number of light-quanta, or photons, of frequency ν radiated per second into the above solid angle according to some observer at rest relative to the star, the rate at which these same photons are received by the observer is N_ν/D . Also, as the observed energy of a photon of observed frequency ν is $h\nu$, the energy received by the observer from a photon radiated by the star at frequency ν is $h\nu/D$. Hence, the total energy received per second by the observer is

$$E = (\sum h\nu N_\nu)/D^2. \quad (1)$$

The observer therefore measures the intensity as E/V where V is the area of his receiving plate, and he defines the distance, Δ' , to be such that $\Delta'^2 E/V$ is the specific intensity of radiation.

It is assumed that the observer can measure the specific intensity by some method, and so find Δ' . To see what this intensity is, we must consider an observer B' at rest relative to the star and sufficiently near the star for him to measure the distance directly. For convenience, let us place this observer so that his world-line intersects the null geodesic C at a point P' . Then if V' is the area of cross-section of the above pencil of null geodesics made by the instantaneous space of B' at P' , this observer receives an amount of energy $\sum h\nu N_\nu$ on an area V' , and if he measures directly the distance, δ , of the star, the specific intensity of radiation is given by

$$I = (\sum h\nu N_\nu)\delta^2/V'. \quad (2)$$

Hence, for Δ' , we have

$$I = (\sum h\nu N_\nu)\delta^2/V' = (\sum h\nu N_\nu)\Delta'^2/VD^2,$$

i.e.

$$\Delta' = D\delta\sqrt{V/V'}, \quad (3)$$

where, in evaluating this expression, we can take the limit $\delta \rightarrow 0$. Omitting the factor D , we get the result obtained by Etherington, which is equivalent to that found by the author in the previous paper.

Etherington has shown that the expression for distance found from the apparent size is equivalent to *

$$\Delta = \frac{\delta}{D} \sqrt{\frac{V}{V'}} \quad (4)$$

in the above notation, so we have the simple relation

$$\Delta' = D^2\Delta \quad (5)$$

connecting the two distances.

3. *Calculation of Distance.*—We shall use for convenience some results of the previous paper with a slight change of notation. Points of the null geodesic C joining P_1, P_0 have the co-ordinates $x^i (i=0, 1, 2, 3)$ as functions of an affine parameter s having the values s_0, s_1 at P_0, P_1 respectively, and the

* We are not using Etherington's notation in the present paper. He wrote Θ for V , θ for V' and Δ' for the distance from apparent size.

tangent vector dx^i/ds at points of C is written λ^i . A unit vector $\lambda_{1|}^i$ orthogonal to λ^i is given at points of C by parallel transport along C , and another unit vector $\lambda_{2|}^i$ is taken to be orthogonal to and independent of λ^i and $\lambda_{1|}^i$. The components of these vectors are functions of s , and if R_{hijk} are the components at points of C of the curvature tensor, we can construct invariants

$$\Gamma_q^p = \Gamma_p^q = -R_{hijk} \lambda^h \lambda^k \lambda_{p|}^i \lambda_{q|}^j, \quad (p, q = 1, 2) \quad (6)$$

which are therefore functions of s .

Consider now the equations

$$\frac{d^2 z^p}{ds^2} - \Gamma_q^p z^q = 0 \quad (p = 1, 2) \quad (7)$$

giving z^1, z^2 in terms of s , and choose two of the four arbitrary constants so that the z 's vanish when $s = s_1$. Then we can write

$$z^p = \phi_q^p \alpha^q \quad (p = 1, 2) \quad (8)$$

where α^1, α^2 are arbitrary constants and the ϕ 's are functions of s and s_1 , vanishing at s_1 . Forming the function Θ of s_0, s_1 , where

$$\Theta^2 = K |\phi_q^p|_{s=s_0}$$

and

$$\frac{1}{K} = \lim_{s \rightarrow s_1} \frac{|\phi_q^p|}{(s - s_1)^2}, \quad (9)$$

it can be shown that Θ depends only on the positions P_0, P_1 , and the choice of the parameter s .

If A^i is the unit vector tangent to the star's world-line at P_1 , and if g_{ij} is the fundamental tensor of the space-time, it was shown that, in the notation of § 2,

$$\lim_{\delta \rightarrow 0} \delta \sqrt{\frac{V}{V'}} = \Theta (g_{ij} A^i \lambda^j)_{s=s_1}. \quad (10)$$

Now it has been shown* that if B^i is the unit vector tangent to the observer's world-line at P_0 , the Doppler effect, D , is given by

$$D = \frac{(g_{ij} A^i \lambda^j)_{s=s_1}}{(g_{ij} B^i \lambda^j)_{s=s_0}}. \quad (11)$$

Hence, writing †

$$\mu = (g_{ij} B^i \lambda^j)_{s=s_0}, \quad (12)$$

we have, from (4),

$$\Delta = \mu \Theta. \quad (13)$$

This, then, is the distance calculated from the apparent size of the star. It depends only on the positions P_0, P_1 and the vector B^i at P_0 , the product $\mu \Theta$ being independent of the choice of s . The other distance, Δ' , can now

* Kermack, Milne and Whittaker, *Proc. Roy. Soc. Edin.*, 53, 39, 1933.

† It is assumed that s is chosen to increase in passing along C from P_1 to P_0 , in which case μ is positive.

be found from (5), (11) and (3), and we see that it depends on both the vectors A^i and B^i , *i.e.* depends on the motions of both star and observer.

4. *Distance and Luminosity in Special Relativity.*—A result found in the previous paper and given at once by the above method is that in space of constant curvature, and in particular in space of special relativity, the *distance invariant*, Θ , is

$$\Theta = s_0 - s_1. \quad (14)$$

Consider the case in special relativity when the star is moving away from the observer with velocity V , the velocity of light being unity. Then, in the observer's frame of reference, the metric is

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2, \quad (15)$$

where t is the observer's proper time, and the world-line of the star is given by

$$t = s_1 + \sigma \cosh \alpha, \quad x = -s_1 + \sigma \sinh \alpha, \quad y = z = 0,$$

where σ is the arc and $\tanh \alpha = V$. When the observer is at the origin $t = 0$, he receives light emitted by the star at the position $\sigma = 0$, and the path of this light is the null geodesic

$$t = s, \quad x = -s, \quad y = z = 0.$$

Substituting in (12) and (11) the values of g_{ij} , λ^i , A^i and B^i , we find

$$\mu = 1, \quad D = \cosh \alpha + \sinh \alpha,$$

and hence by (13), (5) and (14), as $s_0 = 0$,

$$\Delta = -s_1, \quad \Delta' = -s_1 (\cosh \alpha + \sinh \alpha)^2. \quad (16)$$

Assuming that at some period the star was with the observer, the interval of time t between that event and the event when the observer receives the above signal is

$$\tau = -s_1 (\cosh \alpha + \sinh \alpha) / \sinh \alpha.$$

Hence, substituting $\tanh \alpha = V$, (16) can be written

$$\Delta = \frac{\tau V}{1 + V}, \quad \Delta' = \frac{\tau V}{1 - V}. \quad (17)$$

From the definition of the distance Δ' , the intensity measured by the observer is proportional to $1/\Delta'^2 = (1 - V)^2 / V^2 \tau^2$, and this is in agreement with Milne's formula for apparent brightness.*

5. Some of the properties of conformal spaces can be advantageously applied to the calculation of distance, for it can be shown that if a space S' is conformal to the space S , then the null geodesics of S' correspond to null geodesics of S . If C' , C are two corresponding null geodesics, and if the two fundamental tensors are connected by

$$g'_{ij} = f^2 g_{ij} \quad (18)$$

* E. A. Milne, *Zs. für Astrophysik*, 6, 95, 1933.

where f is a function of the co-ordinates, it can be shown that an affine parameter s' of C' can be chosen so that $ds' = f^2 ds$, where s is the parameter of C , the point s' corresponding to the point s . With this choice of s' , the tangent vector of C' at s' is given by

$$\lambda'^i = f^{-2} \lambda^i, \quad (19)$$

where λ^i is the tangent vector of C at s .

It can also be shown that if s'_0, s'_1 are the points of C' corresponding to the points s_0, s_1 of C , and if Θ', Θ are the corresponding distance invariants defined as in § 3, then

$$\Theta' = f_0 f_1 \Theta, \quad (20)$$

where subscripts 0, 1 denote values at the points s_0, s_1 respectively.

6. *Distance in an Expanding Universe.*—A form of space-time including many well-known forms in general relativity has the metric

$$ds^2 = \xi^2 dt'^2 - R^2 (\eta^{-2} dr'^2 + r'^2 d\theta^2 + r'^2 \sin^2 \theta d\phi^2), \quad (21)$$

where ξ, η are some functions of r' and R is a function of t' . Defining new co-ordinates t, r by $dt = dt'/R, r = r'/\xi$, the form (21) becomes

$$ds^2 = f^2 \{ dt^2 - (\gamma^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2) \}, \quad (22)$$

where γ is a function of r , given by $\gamma = (\xi \eta dr/dr')^2$, and f is a function of t and r , given by $f = R\xi$. Hence, by § 5, the distance in space-time of the form (21) can be deduced from the corresponding distance in space-time with the metric

$$ds^2 = dt^2 - (\gamma^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2), \quad (23)$$

where γ is a function of r , and if this solution is known, we can at once find the distance when the metric is given by (22), f now being any function of the co-ordinates. Let us therefore consider the form (23).

This form has spherical symmetry in the "spatial" co-ordinates r, θ, ϕ , and from the geodesic equations we find that any geodesic must lie in a plane through the origin. There is therefore no loss of generality if we consider only those null geodesics for which $\theta = \pi/2$. The remaining geodesic equations can be solved to give

$$\frac{dt}{ds} = 1, \quad \frac{dr}{ds} = \pm \psi \sqrt{\gamma}, \quad \frac{d\phi}{ds} = \frac{h}{r^2}; \quad \psi^2 = 1 - \frac{h^2}{r^2}, \quad (24)$$

where h is a constant which vanishes if the null geodesic passes through the origin $r = 0$. We now have the components of the vector λ^i .

One vector orthogonal to λ^i and given by parallel transport along this null geodesic is found to be

$$\lambda_{1i} \equiv (0, 0, 1/r, 0), \quad (25)$$

and a vector orthogonal to and independent of λ^i and λ_{1i} is

$$\lambda_{2i} \equiv (h/r\psi, 0, 0, 1/r\psi). \quad (26)$$

Calculating from (23) the components of the curvature tensor, and substituting in (6), we find $\Gamma_2^1 = \Gamma_1^2 = 0$, and by using (24), the remaining invariants can be written in the form

$$\left. \begin{aligned} \Gamma_1^1 &= \frac{1}{r} \frac{d^2 r}{ds^2} - \frac{h^2}{r^4}, \\ \Gamma_2^2 &= \frac{1}{r} \frac{d^2 r}{ds^2} + \frac{h^2}{r^4} \cdot \frac{1}{\psi} \frac{d^2 \psi}{ds^2} \end{aligned} \right\} \quad (27)$$

I. $h \neq 0$.

Substituting in (7), and writing $z^1 = rx$, $z^2 = ry$, we get

$$\frac{d^2 x}{d\phi^2} + x = 0$$

and

$$\frac{d^2 y}{d\phi^2} - \frac{y}{\psi} \frac{d^2 \psi}{d\phi^2} = 0.$$

Hence, the required solutions are

$$z^1 = a^1 r \sin(\phi - \phi_1), \quad z^2 = a^2 r \psi(U - U_1), \quad (28)$$

where

$$U = \int \frac{d\phi}{\psi^2} \quad (29)$$

and subscripts 0, 1 denote values at the points s_0 , s_1 respectively. We therefore find, from (9),

$$\Theta^2 = Kr_0^2 \psi_0(U_0 - U_1) \sin(\phi_0 - \phi_1),$$

where

$$\frac{1}{K} = \left(r^2 \psi \frac{dU}{ds} \frac{d\phi}{ds} \right)_{s=s_1},$$

i.e., from (24),

$$K = r_1^2 \psi_1 / h^2.$$

Hence, we have

$$\Theta = \frac{1}{h} r_0 r_1 \{ \psi_0 \psi_1 (U_0 - U_1) \sin(\phi_0 - \phi_1) \}^{1/2}. \quad (30)$$

II. $h = 0$.

In this case

$$\Gamma_1^1 = \Gamma_2^2 = \frac{1}{r} \frac{d^2 r}{ds^2}.$$

Substituting in (7), the equations can be solved, and we find

$$\Theta = r_0 r_1 |W_0 - W_1|, \quad (31)$$

where

$$W = \int^r \frac{1}{\sqrt{\gamma}} \frac{dr}{r^2}. \quad (32)$$

Returning now to the more general form (22), then from (19) a typical null geodesic is given by $\theta = \pi/2$ and

$$\frac{dt}{ds} = \frac{1}{f^2}, \quad \frac{dr}{ds} = \pm \frac{\psi \sqrt{\gamma}}{f^2}, \quad \frac{d\phi}{ds} = \frac{h}{f^2 r^2}; \quad \psi^2 = 1 - \frac{h^2}{r^2}. \quad (33)$$

The new invariant Θ for two points s_0, s_1 of this geodesic can be found by using (20). Hence from (30) and (13), the distance when $h \neq 0$ is

$$\Delta = \frac{\mu}{h} f_0 f_1 r_0 r_1 \{\psi_0 \psi_1 (U_0 - U_1) \sin(\phi_0 - \phi_1)\}^{1/2}, \quad (34)$$

where μ is given by (12) and U by (29). When $h = 0$, then from (31) the distance is

$$\Delta = \mu f_0 f_1 r_0 r_1 |W_0 - W_1| \quad (35)$$

where W is given by (32).

A case of particular interest is when the observer is at the origin. Here $r_0 = 0$, and we find in general

$$\lim_{r \rightarrow 0} (rW) = -\frac{1}{\sqrt{\gamma_0}},$$

where the subscript 0 now denotes values at the origin $r = 0$. Hence, from (35), for a star at the position r ,

$$\Delta = \mu (f/\sqrt{\gamma})_0 fr. \quad (36)$$

If we assume that the observer is instantaneously "at rest" with respect to the spatial co-ordinates, the vector B^i is

$$B^i \equiv (1/f_0, 0, 0, 0),$$

and from (12), (33) and (22)

$$\mu = 1/f_0.$$

Hence, from (36),

$$\Delta = \gamma_0^{-1/2} fr. \quad (37)$$

7. *The Modified Lemaître Universe.*—The form for a homogeneous isotropic universe can be written *

$$ds^2 = R^2 \{ dt^2 - (\gamma^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2) \}, \quad \gamma = 1 - kr^2, \quad (38)$$

where $k = 1, 0$, or -1 , and R is a function of t only. Comparing this form with (22), we see from (33) that the path of light received by an observer at the origin is given by

$$dr/dt = -\sqrt{1 - kr^2}, \quad (39)$$

and the distance of a star from the origin is, by (37),

$$\Delta = Rr. \quad (40)$$

* H. P. Robertson, *Reviews of Modern Physics*, 5, 62, 1932. For convenience, we have written Rdt for the usual element dt .

If the star is assumed to be at rest with respect to the spatial co-ordinates, the Doppler effect can be found from (11), and we get $D = R_0/R$. Define an observable, v , by

$$v = D - 1 = \frac{R_0}{R} - 1. \quad (41)$$

Then for small values, v is the apparent velocity of recession of the star.* Assuming that R is known, we can by (39) and (41) expand r and R in terms of v and substitute in (40) to obtain a relation between Δ and v . To the required order, we can write this relation in the form

$$v/\Delta = K_1 + K_2 v, \quad (42)$$

which is an extension of Hubble's Law $v/\Delta = \text{const.}$ As v and Δ are observables, K_1 and K_2 are observable constants. They are given in terms of R by

$$K_1 = R_0'/R_0^2, \quad K_2 = (4R_0'^2 - R_0 R_0'')/2R_0' R_0^2, \quad (43)$$

where dashes denote differentiations with respect to t , and the subscript denotes evaluation at the time of observation, *i.e.* at the origin. These equations therefore give R_0' and R_0'' in terms of R_0 .

In addition to K_1 and K_2 , there are two other observable constants K_3, K_4 , given by

$$K_3 = 2\pi\delta_0, \quad K_4 = 8\pi p_0, \quad (44)$$

where p is the pressure and δ the proper density. It can be shown from (38) that

$$\left. \begin{aligned} 2\pi\delta &= -\lambda + 3(kR + R'')/2R^3, \\ 8\pi p &= \lambda - (kR^2 - R'^2 + 2RR'')/R^4, \end{aligned} \right\} \quad (45)$$

where λ is the cosmical constant at present undetermined. Evaluating at the origin, these equations with (44) give two relations between R_0, R_0', R_0'' and λ .

Eliminating R_0', R_0'' and λ from these relations and (43), we find

$$k/2R_0^2 = K_1(K_1 - K_2) + K_3 + K_4. \quad (46)$$

Hence, if

$$K_1(K_1 - K_2) + K_3 + K_4 >, =, < 0, \quad k = 1, 0, -1. \quad (47)$$

Thus k can be determined, and if $k \neq 0$, R_0, R_0' and R_0'' can be found from (46) and (43). Substituting in either of (45), we get

$$\lambda = 9K_1^2 - 6K_1K_2 + 2K_3 + 3K_4. \quad (48)$$

If $k = 0$, R_0 cannot be determined, but we find that λ is still given by (48). From (47), this can now be written

$$\lambda = 3K_1^2 - 4K_3 - 3K_4. \quad (49)$$

* From special relativity, we should say that the actual velocity of recession, V , is given by $(1 + V)/\sqrt{1 - V^2} = D = 1 + v$. Thus v can take any values, but V cannot exceed unity.

From observations, K_3 and K_4 are found to be very small compared with K_1^2 , and may therefore be neglected in (48) and (49). Thus if $k=0$, λ is positive. If $k=1$, then from (47) and (48) $\lambda > 3K_1^2$, i.e. λ is positive. Hence, if $k=1$ or 0 , λ is positive, and if $k=-1$, λ may be positive, negative, or zero.

These results show that k and λ can be completely determined, and are not at our disposal as some writers have indicated. It must be remembered that Δ in (42) is necessarily the distance derived from apparent size, and from distances Δ' by apparent brightness Δ must be found by (5).

Merton College, Oxford :
1933 October.

ON THE PERIODIC ORBITS IN THE NEIGHBOURHOOD OF THE TRIANGULAR EQUILIBRIUM POINTS IN THE RESTRICTED PROBLEM OF THREE BODIES.

Peder Pedersen.

(Communicated by E. Strömgren)

Introduction.—Since Gylden* in 1884 began the study of the infinitesimal periodic orbits around the libration points in the restricted problem of three bodies, a number of investigations of the problem have appeared. Most of these investigations are based on the variational equations that correspond to the differential equations of the problem. As E. Strömgren† has shown, the variational equations are suitable as a basis for numerical integrations. In theoretical investigations, however, essential peculiarities of the periodic orbits are often lost when the variational equations are used.

The following example may illustrate the question: Using the variational equations one finds that the period of the periodic orbits around a libration point is a function of the two finite masses, while it is independent of the size of the orbit. Using the differential equations, however, one finds that the period must depend on the size of the orbit. Due to the fact that only terms of the first order are retained in forming the variational equations, the essential property of the orbits that the period varies with the size is lost. When terms of higher order are retained in the analysis the dependence of period on size reappears.

In the present investigation I will show that the period of the periodic orbits around the libration points L_4 and L_5 appear as a function of the size of the orbits, when terms of *second and third order* are retained in the series expressions, limiting, however, the investigation to the case that the masses have values in the *neighbourhood* of the critical values μ_0 and $1 - \mu_0$. It will further be shown that the critical values of the masses depend on the

* "Sur un cas particulier du problème des trois corps," *Bull. Astr.*, I, 361-369.

† *Publikationer fra Københavns Observatorium.*

size of the orbits in such a way that the critical value corresponding to the small mass increases with the size of the orbit. Finally, it will be shown that classes of periodic orbits with *infinitesimal limiting orbits* exist for values of the small mass greater than the critical value μ_0 .

I am indebted to Professor Elis Strömgren for some valuable suggestions.

§ 1. We shall begin with fixing the units of length, mass and time. As unit of length we choose the distance between the two finite masses, as unit of mass the sum of the two finite masses, and, finally, the unit of time is determined by putting the angular velocity of the two finite masses equal to 1. With this choice of units the gravitational constant reduces to 1.

It should be mentioned that these units which are convenient in theoretical investigations are not identical with the units adopted in the so-called Copenhagen problem.

We shall call the values of the two finite masses μ and $1 - \mu$, assuming $\mu \leq \frac{1}{2}$. The motion of the three masses (μ , $1 - \mu$ and the infinitesimal mass) will be referred to a rotating co-ordinate system, the origin of which is the centre of mass of the two finite masses; the x -axis passes through the finite masses, the positive direction being that from the origin to μ . The positive direction of revolution is chosen as the direction of the absolute motion of the two finite masses. In this co-ordinate system the co-ordinates of μ and $1 - \mu$ are $(1 - \mu, 0)$ and $(-\mu, 0)$.

The differential equations of motion of the infinitesimal mass are :

$$\begin{aligned} \ddot{x} - 2\dot{y} - x &= \frac{\partial \Omega}{\partial x}, \\ \ddot{y} + 2\dot{x} - y &= \frac{\partial \Omega}{\partial y}, \end{aligned} \quad (1)$$

where

$$\Omega = \frac{1 - \mu}{\rho_1} + \frac{\mu}{\rho_2}, \quad (2)$$

and ρ_1 and ρ_2 are determined by the equations

$$\begin{aligned} \rho_1^2 &= (x + \mu)^2 + y^2, \\ \rho_2^2 &= (x + \mu - 1)^2 + y^2. \end{aligned} \quad (3)$$

Fig. 1 shows the positions of the libration points when μ has the critical value μ_0 ($\mu_0 = 0.03852 \dots$; μ_0 is determined by the equation $\mu_0(1 - \mu_0) = \frac{1}{27}$).

L_4 and L_5 have the co-ordinates $(\frac{1}{2} - \mu, \frac{1}{2}\sqrt{3})$ and $(\frac{1}{2} - \mu, -\frac{1}{2}\sqrt{3})$.

To investigate the motion of the infinitesimal mass in the neighbourhood of the libration point L_4 we introduce a new co-ordinate system with its origin in L_4 and with axis parallel to the axis of the former co-ordinate system. We then have

$$\begin{aligned} x &= \xi + \frac{1}{2} - \mu, \\ y &= \eta + \frac{1}{2}\sqrt{3}, \end{aligned} \quad (4)$$

hence the differential equations of motion of the infinitesimal mass in the new co-ordinate system are :

$$\begin{aligned}\ddot{\xi} - 2\dot{\eta} - \xi - \frac{1}{2} + \mu &= \frac{\partial \Omega}{\partial \xi}, \\ \ddot{\eta} + 2\dot{\xi} - \eta - \frac{1}{2}\sqrt{3} &= \frac{\partial \Omega}{\partial \eta},\end{aligned}\quad (5)$$

where

$$\Omega = \frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2}, \quad (6)$$

and where ρ_1 and ρ_2 are determined by the equations

$$\begin{aligned}\rho_1^2 &= (\xi + \frac{1}{2})^2 + (\eta + \frac{1}{2}\sqrt{3})^2, \\ \rho_2^2 &= (\xi - \frac{1}{2})^2 + (\eta + \frac{1}{2}\sqrt{3})^2.\end{aligned}\quad (7)$$

The differential equations possess the Jacobian integral

$$\dot{\xi}^2 + \dot{\eta}^2 = \xi^2 + \eta^2 + (1-2\mu)\xi + \sqrt{3}\eta + 2\Omega - C, \quad (8)$$

where C is the Jacobian constant.

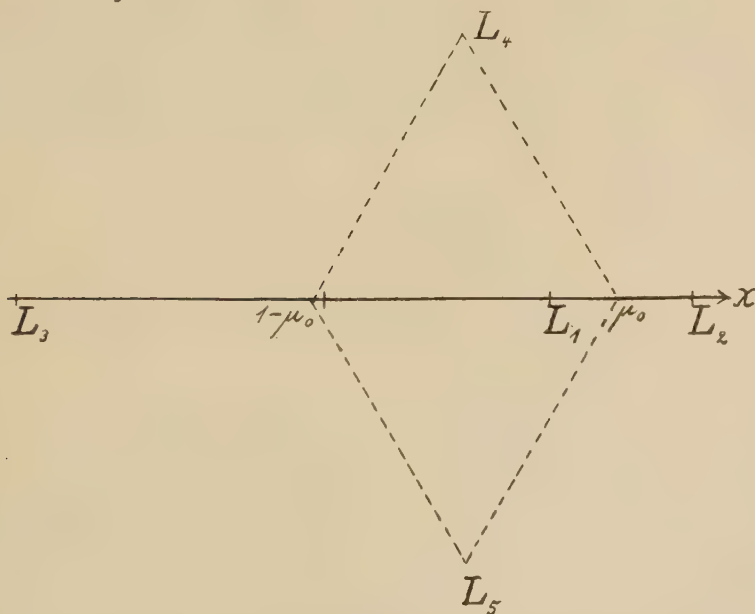


FIG. 1.

§ 2. The above-mentioned critical value of the mass μ ($\mu_0 = 0.03852 \dots$) has the well-known property that infinitesimal periodic orbits exist around the libration points L_4 and L_5 when $\mu \leq \mu_0$, while no such orbits exist for $\mu > \mu_0$. Infinitesimal periodic orbits around the libration points L_1 , L_2 and L_3 exist for all values of μ .

The critical value μ_0 is, however, determined, not from the differential equations (1) or (5), but from variational equations corresponding to (5) and obtained from (5) by retaining terms of the first order only. The

critical value μ_0 can therefore be said to correspond to the variational equations, and as the variational equations are only a first approximation to the differential equations, similarly μ_0 is only a first approximation to the critical mass corresponding to (5).

In the following we shall show how the critical value of the mass can be determined as a function of the size of the periodic orbit by retaining terms up to the *third order* in the series expressions. Only in the limiting case that the dimension of the orbit is zero the critical mass has the value μ_0 .

§ 3. Now let us assume that the infinitesimal mass, the motion of which we want to examine, is situated in the immediate neighbourhood of the libration point L_4 . We shall assume its co-ordinates ξ and η to be small quantities of the first order, and similarly we shall assume that $\dot{\xi}$ and $\dot{\eta}$ are small quantities of the first order. With the aid of the differential equations (5) we then see at once that $\ddot{\xi}$ and $\ddot{\eta}$ are also small quantities of the first order.

We now expand the two terms on the right-hand sides of the equations (5), $\frac{\partial \Omega}{\partial \xi}$ and $\frac{\partial \Omega}{\partial \eta}$, in series and retain terms up to the third order. In the expansion of Ω it will then be necessary to retain terms up to the fourth order, as the order of the terms is diminished by 1 by the process of differentiation.

We start with expanding $\frac{1}{\rho_1}$ and $\frac{1}{\rho_2}$, then find the series expression for Ω and finally for $\frac{\partial \Omega}{\partial \xi}$ and $\frac{\partial \Omega}{\partial \eta}$.

Equations (7) can be written in the following form :—

$$\begin{aligned}\rho_1^2 &= 1 + \xi + \sqrt{3}\eta + \xi^2 + \eta^2, \\ \rho_2^2 &= 1 - \xi + \sqrt{3}\eta + \xi^2 + \eta^2.\end{aligned}\quad (9)$$

From these equations we get

$$\begin{aligned}\frac{1}{\rho_1} &= 1 - \frac{1}{2}\xi - \frac{1}{2}\sqrt{3}\eta - \frac{1}{8}\xi^2 + \frac{3}{4}\sqrt{3}\xi\eta + \frac{5}{8}\eta^2 \\ &\quad + \frac{7}{128}\xi^3 - \frac{3}{16}\sqrt{3}\xi^2\eta - \frac{3}{16}\xi\eta^2 - \frac{3}{16}\sqrt{3}\eta^3 \\ &\quad - \frac{3}{128}\xi^4 - \frac{2}{32}\sqrt{3}\xi^3\eta + \frac{1}{64}\xi^2\eta^2 + \frac{4}{32}\sqrt{3}\xi\eta^3 + \frac{3}{128}\eta^4, \\ \frac{1}{\rho_2} &= 1 + \frac{1}{2}\xi - \frac{1}{2}\sqrt{3}\eta - \frac{1}{8}\xi^2 - \frac{3}{4}\sqrt{3}\xi\eta + \frac{5}{8}\eta^2 \\ &\quad - \frac{7}{128}\xi^3 - \frac{3}{16}\sqrt{3}\xi^2\eta + \frac{3}{16}\xi\eta^2 - \frac{3}{16}\sqrt{3}\eta^3 \\ &\quad - \frac{3}{128}\xi^4 + \frac{2}{32}\sqrt{3}\xi^3\eta + \frac{1}{64}\xi^2\eta^2 - \frac{4}{32}\sqrt{3}\xi\eta^3 + \frac{3}{128}\eta^4.\end{aligned}\quad (10)$$

Inserting in (6) we next find the expansion of Ω :

$$\begin{aligned}\Omega &= 1 - \frac{1}{2}(1 - 2\mu)\xi - \frac{1}{2}\sqrt{3}\eta \\ &\quad - \frac{1}{8}\xi^2 + \frac{3}{4}\sqrt{3}(1 - 2\mu)\xi\eta + \frac{5}{8}\eta^2 \\ &\quad + \frac{7}{128}(1 - 2\mu)\xi^3 - \frac{3}{16}\sqrt{3}(1 - 2\mu)\xi^2\eta - \frac{3}{16}(1 - 2\mu)\xi\eta^2 - \frac{3}{16}\sqrt{3}\eta^3 \\ &\quad - \frac{3}{128}\xi^4 - \frac{2}{32}\sqrt{3}(1 - 2\mu)\xi^3\eta + \frac{1}{64}\xi^2\eta^2 + \frac{4}{32}\sqrt{3}(1 - 2\mu)\xi\eta^3 + \frac{3}{128}\eta^4.\end{aligned}\quad (11)$$

From (11) we finally obtain the expansions of $\frac{\partial \Omega}{\partial \xi}$ and $\frac{\partial \Omega}{\partial \eta}$. These are then inserted in (5) and the following differential equations, on which the following analysis is based, are obtained:—

$$\begin{aligned} \ddot{\xi} - 2\dot{\eta} - \frac{3}{4}\xi - \frac{3}{4}\sqrt{3}(1-2\mu)\eta \\ - \frac{2}{16}(1-2\mu)\xi^2 + \frac{3}{8}\sqrt{3}\xi\eta + \frac{3}{16}(1-2\mu)\eta^2 \\ + \frac{3}{8}\xi^3 + \frac{7}{32}\sqrt{3}(1-2\mu)\xi^2\eta - \frac{1}{32}\frac{3}{2}\xi\eta^2 - \frac{4}{32}\sqrt{3}(1-2\mu)\eta^3 = 0, \end{aligned} \quad (12a)$$

$$\begin{aligned} \ddot{\eta} + 2\dot{\xi} - \frac{3}{4}\sqrt{3}(1-2\mu)\xi - \frac{3}{4}\eta \\ + \frac{3}{16}\sqrt{3}\xi^2 + \frac{3}{8}(1-2\mu)\xi\eta + \frac{9}{16}\sqrt{3}\eta^2 \\ + \frac{2}{32}\sqrt{3}(1-2\mu)\xi^3 - \frac{1}{32}\frac{3}{2}\xi^2\eta - \frac{1}{32}\frac{5}{2}\sqrt{3}(1-2\mu)\xi\eta^2 - \frac{3}{32}\eta^3 = 0. \end{aligned} \quad (12b)$$

The Jacobian integral corresponding to (12) is obtained from (8):

$$\begin{aligned} \xi^2 + \eta^2 = 2 + \frac{3}{4}\xi^2 + \frac{3}{2}\sqrt{3}(1-2\mu)\xi\eta + \frac{9}{4}\eta^2 \\ + \frac{7}{8}(1-2\mu)\xi^3 - \frac{3}{8}\sqrt{3}\xi^2\eta - \frac{3}{8}(1-2\mu)\xi\eta^2 - \frac{3}{8}\sqrt{3}\eta^3 \\ - \frac{3}{8}\frac{7}{4}\xi^4 - \frac{2}{16}\sqrt{3}(1-2\mu)\xi^3\eta + \frac{1}{32}\frac{3}{2}\xi^2\eta^2 + \frac{4}{16}\sqrt{3}(1-2\mu)\xi\eta^3 + \frac{3}{8}\eta^4 - C, \end{aligned} \quad (13)$$

where the Jacobian constant has the same value as in (8).

§ 4. Finding the periodic orbits around L_4 is equivalent to determining the coefficients a and b in the Fourier expansions of ξ and η :

$$\begin{aligned} \xi = a_0 + a_1 \overset{\text{II}}{\cos \omega t} + a_{-1} \overset{\text{I}}{\sin \omega t} + a_2 \overset{\text{II}}{\cos 2\omega t} + a_{-2} \overset{\text{II}}{\sin 2\omega t} \\ + a_3 \overset{\text{III}}{\cos 3\omega t} + a_{-3} \overset{\text{III}}{\sin 3\omega t}, \\ \eta = b_0 + b_1 \overset{\text{II}}{\cos \omega t} + b_{-1} \overset{\text{I}}{\sin \omega t} + b_2 \overset{\text{II}}{\cos 2\omega t} + b_{-2} \overset{\text{II}}{\sin 2\omega t} \\ + b_3 \overset{\text{III}}{\cos 3\omega t} + b_{-3} \overset{\text{III}}{\sin 3\omega t}. \end{aligned} \quad (14)$$

The roman numbers denote the order of the coefficients. An examination of the variational equations shows that a_1 , a_{-1} , b_1 and b_{-1} are of the first order, while the other coefficients are of higher order. We assume that the coefficients with the indices 0 and ± 2 are of the second order, that the coefficients with indices ± 3 are of the third order and that the coefficients with indices numerically greater than 3 are of higher order than the third. As only terms up to the third order are retained in the expansions for ξ and η , we break off the Fourier expansion after the first seven terms. We then have to determine seven coefficients a and seven coefficients b by the condition that ξ and η given by expressions (14) shall satisfy (12).

The first step is forming the Fourier expansions of the quantities of the first order $\dot{\xi}$, $\dot{\eta}$, $\ddot{\xi}$ and $\ddot{\eta}$, of the quantities of the second order ξ^2 , $\xi\eta$ and η^2 , and of the quantities of the third order ξ^3 , $\xi^2\eta$, $\xi\eta^2$ and η^3 ; terms up to the third order are retained. For greater clearness the Fourier coefficients are arranged in schematical form.

The next step is the determination of the four coefficients a_1 , a_{-1} , b_1

Coefficient Scheme

	1	$\cos \omega t$	$\sin \omega t$	$\cos 2\omega t$	$\sin 2\omega t$	$\cos 3\omega t$	$\sin 3\omega t$
ξ	a_0	a_1	a_{-1}	a_2	a_{-2}	a_3	a_{-3}
η	b_0	b_1	b_{-1}	b_2	b_{-2}	b_3	b_{-3}
ξ^2	$\dots$	ωa_{-1}	$-\omega a_1$	$2\omega a_{-2}$	$-2\omega a_2$	$3\omega a_{-3}$	$-3\omega a_3$
η^2	$\dots$	ωb_{-1}	$-\omega b_1$	$2\omega b_{-2}$	$-2\omega b_2$	$3\omega b_{-3}$	$-3\omega b_3$
ξ^3	$\dots$	$-\omega^2 a_1$	$-\omega^2 a_{-1}$	$-4\omega^2 a_2$	$-4\omega^2 a_{-2}$	$-9\omega^2 a_3$	$-9\omega^2 a_{-3}$
η^3	$\dots$	$-\omega^2 b_1$	$-\omega^2 b_{-1}$	$-4\omega^2 b_2$	$-4\omega^2 b_{-2}$	$-9\omega^2 b_3$	$-9\omega^2 b_{-3}$
ξ^2	$\frac{1}{2}a_1^2 + \frac{1}{2}a_{-1}^2$	$2a_0a_1$	$2a_0a_{-1}$	$\frac{1}{2}a_1^2 - \frac{1}{2}a_{-1}^2$	a_1a_{-1}	$a_1a_2 - a_{-1}a_{-2}$	$a_1a_{-2} + a_{-1}a_2$
$\xi\eta$	$\frac{1}{2}a_1b_1 + \frac{1}{2}a_{-1}b_{-1}$	$\frac{1}{2}(a_1b_2 + a_2b_1)$	$\frac{1}{2}(a_1b_{-2} + a_{-2}b_1)$	$\frac{1}{2}a_1b_1 - \frac{1}{2}a_{-1}b_{-1}$	$\frac{1}{2}a_1b_{-1} + \frac{1}{2}a_{-1}b_1$	$\frac{1}{2}(a_1b_2 + a_2b_1)$	$\frac{1}{2}(a_1b_{-2} + a_{-2}b_1)$
η^2	$\frac{1}{2}b_1^2 + \frac{1}{2}b_{-1}^2$	$2b_0b_1$	$2b_0b_{-1}$	$\frac{1}{2}b_1^2 - \frac{1}{2}b_{-1}^2$	b_1b_{-1}	$b_1b_2 - b_{-1}b_{-2}$	$b_1b_{-2} + b_{-1}b_2$
ξ^3	$\dots$	$\frac{3}{2}a_1^3 + \frac{3}{2}a_1a_{-1}^2$	$\frac{3}{2}a_{-1}^3 + \frac{3}{2}a_1^2a_{-1}$	$\dots$	$\dots$	$\frac{1}{4}a_1^3 - \frac{3}{4}a_1a_{-1}^2$	$-\frac{1}{4}a_{-1}^3 + \frac{3}{4}a_1^2a_{-1}$
$\xi^2\eta$	$\dots$	$\frac{3}{2}a_1^2b_1$	$\frac{3}{2}a_1^2b_{-1}$	$\dots$	$\dots$	$\frac{1}{4}a_1^2b_1$	$\frac{1}{4}a_1^2b_{-1}$
$\xi\eta^2$	$\dots$	$\frac{3}{2}a_1b_1^2$	$\frac{3}{2}a_1b_{-1}^2$	$\dots$	$\dots$	$-\frac{1}{4}a_1^2b_{-1}$	$-\frac{1}{4}a_1^2b_1$
$\xi\eta^3$	$\dots$	$\frac{3}{2}a_1b_1^3$	$\frac{3}{2}a_1b_{-1}^3$	$\dots$	$\dots$	$-\frac{1}{2}a_1b_1b_{-1}$	$-\frac{1}{2}a_1b_{-1}b_1$
η^3	$\dots$	$\frac{3}{2}b_1^3 + \frac{3}{2}b_1b_{-1}^2$	$\frac{3}{2}b_{-1}^3 + \frac{3}{2}b_1^2b_{-1}$	$\dots$	$\dots$	$\frac{1}{4}b_1^3 - \frac{3}{4}b_1b_{-1}^2$	$-\frac{1}{4}b_{-1}^3 + \frac{3}{4}b_1^2b_{-1}$

and b_{-1} , terms of the first order only being retained. In the differential equations

$$\begin{aligned}\ddot{\xi} - 2\dot{\eta} - \frac{3}{4}\xi - \frac{3}{4}\sqrt{3}(1-2\mu)\eta &= 0, \\ \ddot{\eta} + 2\dot{\xi} - \frac{3}{4}\sqrt{3}(1-2\mu)\xi - \frac{9}{4}\eta &= 0,\end{aligned}\quad (15)$$

which are obtained from (12) when only terms of the first order are retained, we insert

$$\begin{aligned}\xi &= a_1 \cos \omega t + a_{-1} \sin \omega t, \\ \eta &= b_1 \cos \omega t + b_{-1} \sin \omega t.\end{aligned}\quad (16)$$

Equating the coefficients of $\cos \omega t$ and $\sin \omega t$ in the two equations (15) to zero, we get the following system of four equations:—

$$\left. \begin{aligned}(\omega^2 + \frac{3}{4}) \cdot a_1 + \frac{3}{4}\sqrt{3}(1-2\mu) \cdot b_1 &+ 2\omega \cdot b_{-1} = 0, \\ \frac{3}{4}\sqrt{3}(1-2\mu) \cdot a_1 + (\omega^2 + \frac{9}{4}) \cdot b_1 &- 2\omega \cdot a_{-1} = 0, \\ -2\omega \cdot b_1 + (\omega^2 + \frac{3}{4}) \cdot a_{-1} + \frac{3}{4}\sqrt{3}(1-2\mu) \cdot b_{-1} &= 0, \\ 2\omega \cdot a_1 + \frac{3}{4}\sqrt{3}(1-2\mu) \cdot a_{-1} + (\omega^2 + \frac{9}{4}) \cdot b_{-1} &= 0.\end{aligned} \right\} \quad (17)$$

From the two first equations (17) we can express a_{-1} and b_{-1} in terms of a_1 and b_1 :

$$\begin{aligned}2\omega \cdot a_{-1} &= \frac{3}{4}\sqrt{3}(1-2\mu) \cdot a_1 + (\omega^2 + \frac{9}{4}) \cdot b_1, \\ 2\omega \cdot b_{-1} &= -(\omega^2 + \frac{3}{4}) \cdot a_1 - \frac{3}{4}\sqrt{3}(1-2\mu) \cdot b_1.\end{aligned}\quad (18)$$

Inserting these expressions for a_{-1} and b_{-1} into the two last equations (17), we get two equations which determine a_1 and b_1 :

$$\begin{aligned}[\omega^4 - \omega^2 + \frac{2}{15} - \frac{2}{15}(1-2\mu)^2] \cdot a_1 &= 0, \\ [\omega^4 - \omega^2 + \frac{2}{15} - \frac{2}{15}(1-2\mu)^2] \cdot b_1 &= 0.\end{aligned}\quad (19)$$

If now

$$\omega^4 - \omega^2 + \frac{2}{15} - \frac{2}{15}(1-2\mu)^2 \geq 0, \quad (20)$$

then we have

$$a_1 = b_1 = a_{-1} = b_{-1} = 0. \quad (21)$$

The mathematical condition for infinitesimal periodic orbits therefore is

$$\omega^4 - \omega^2 + \frac{2}{15} - \frac{2}{15}(1-2\mu)^2 = 0. \quad (22)$$

A further condition is that ω^2 should be real and positive.

From (22) we get

$$\omega^2 = \frac{1}{2} \pm \sqrt{1 - 27\mu(1-\mu)}. \quad (23)$$

The condition for ω^2 being real is

$$\mu(1-\mu) \leq \frac{1}{27} \quad (24)$$

or the product of the two finite masses should be less than or equal to $\frac{1}{27}$.

The value of the small mass that correspond to the mass-product $\frac{1}{27}$ has already been introduced and denoted by μ_0 . Hence we have

$$\mu_0(1-\mu_0) = \frac{1}{27}. \quad (25)$$

We now assume that the values of the finite masses satisfy (24). From (22) we can then determine ω in such a way that equations (19) are satisfied

for all values of a_1 and b_1 , the last-mentioned quantities being therefore arbitrary constants of integration. Having chosen a_1 and b_1 we can find a_{-1} and b_{-1} from equations (18).

It is easily seen, however, that one of the arbitrary constants is of no importance when the form of the periodic orbit only is of interest. The two equations :

$$\begin{aligned}\xi &= a_1 \cos \omega(t - t_0) + a_{-1} \sin \omega(t - t_0), \\ \eta &= b_1 \cos \omega(t - t_0) + b_{-1} \sin \omega(t - t_0),\end{aligned}\quad (26)$$

correspond to the same motion as the two equations (16), but rearranging the expressions on the right-hand sides of (26) we immediately see that other coefficients of $\cos \omega t$ and $\sin \omega t$ are obtained. The arbitrary constant t_0 , which may vary without changing the orbital motion, we shall call the phase-constant. It is possible to choose the phase-constant in such a way that the coefficient of $\cos \omega t$ in the equation

$$\eta = b_1 \cos \omega(t - t_0) + b_{-1} \sin \omega(t - t_0) \quad (27)$$

vanishes ; t_0 then has to satisfy the equation

$$b_1 \cos \omega t_0 - b_{-1} \sin \omega t_0 = 0. \quad (28)$$

In other words, we can choose the phase-constant in such a way that no terms with $\cos \omega t$ appear in the expansion of η .

We now return to the two equations (16) in which we have considered a_1 and b_1 to be arbitrary constants. Let us assume that the phase-constant has been so fixed that

$$b_1 = 0. \quad (29)$$

In the equations there remains the arbitrary constant a_1 , which we shall call the orbit-constant. The quantities a_{-1} and b_{-1} can be found expressed by a_1 with the aid of (18). As ξ and η have been assumed to be small quantities of the first order, a_1 , too, must be of the first order. In the following we shall denote a_1 by ϵ :

$$a_1 = \epsilon, \quad (30)$$

where ϵ is a small quantity of the first order.

Introducing (29) and (30) in (18) we get two equations for determining a_{-1} and b_{-1} :

$$\begin{aligned}2\omega \cdot a_{-1} &= \frac{3}{4} \sqrt{3} (1 - 2\mu) \epsilon, \\ 2\omega \cdot b_{-1} &= -(\omega^2 + \frac{3}{4}) \epsilon.\end{aligned}\quad (31)$$

We can summarize the above discussion as follows : When

$$\mu(1 - \mu) \leq \frac{1}{2}\tau$$

infinitesimal periodic orbits around L_4 exist, corresponding to the following equations :—

$$\begin{aligned}\xi &= \epsilon \cos \omega t + \frac{3}{8\omega} \sqrt{3} (1 - 2\mu) \epsilon \sin \omega t, \\ \eta &= -\left(\frac{1}{2}\omega + \frac{3}{8\omega}\right) \epsilon \sin \omega t.\end{aligned}\quad (32)$$

§ 5. We shall now give a short account of the method by which the other Fourier coefficients in the Fourier expressions for ξ and η are determined.

Using the values of a_1 , a_{-1} , b_1 and b_{-1} just found :

$$\begin{aligned} a_1 &= \epsilon, & a_{-1} &= \frac{3}{8\omega} \sqrt{3}(1-2\mu)\epsilon, \\ b_1 &= 0, & b_{-1} &= -\left(\frac{1}{2}\omega + \frac{3}{8\omega}\right)\epsilon, \end{aligned} \quad (33)$$

we can compute the coefficients of the second order in the first column of the coefficient scheme corresponding to ξ^2 , $\xi\eta$ and η^2 . Introducing now the values found, together with a_0 and b_0 in the differential equations (12), we can find the constant terms in the Fourier expressions (12a) and (12b). Putting these terms equal to zero two equations of the first order are found for the determination of a_0 and b_0 . As the determinant corresponding to these equations does not vanish, a_0 and b_0 are determined as functions of ϵ .

In a similar way it is possible to compute the coefficients of the second order in the coefficient scheme corresponding to $\cos 2\omega t$ and $\sin 2\omega t$ and ξ^2 , $\xi\eta$ and η^2 . Introducing the values thus found, together with a_2 , a_{-2} , b_2 and b_{-2} in the differential equations (12), we can determine the coefficients of $\cos 2\omega t$ and $\sin 2\omega t$ in the Fourier expansions of (12a) and (12b). Equating these four coefficients to zero, we obtain four equations of the first order for the determination of a_2 , a_{-2} , b_2 and b_{-2} . As the determinant corresponding to these four equations does not vanish, a_2 , a_{-2} , b_2 and b_{-2} can be found as functions of ϵ .

With the values of a_0 , b_0 , a_2 , a_{-2} , b_2 and b_{-2} thus found we can compute the coefficients of the third order in the coefficient scheme corresponding to $\cos \omega t$ and $\sin \omega t$ and ξ^2 , $\xi\eta$, η^2 , ξ^3 , $\xi^2\eta$, $\xi\eta^2$ and η^3 . Introducing in the differential equations the values found together with a_1 , a_{-1} , b_1 and b_{-1} corresponding to ξ , η , $\dot{\xi}$, $\dot{\eta}$, $\ddot{\xi}$ and $\ddot{\eta}$ and considered as unknown, we can determine the coefficients of $\cos \omega t$ and $\sin \omega t$ in the Fourier expressions of (12a) and (12b). Equating these four coefficients to zero, we get for the more exact determination of a_1 , a_{-1} , b_1 and b_{-1} four equations of the form :

$$\left. \begin{aligned} -(\omega^2 + \frac{3}{4}) \cdot a_1 - \frac{3}{4} \sqrt{3}(1-2\mu) \cdot b_1 & - 2\omega \cdot b_{-1} + q_1 \cdot \epsilon = 0, \\ -\frac{3}{4} \sqrt{3}(1-2\mu) \cdot a_1 - (\omega^2 + \frac{9}{4}) \cdot b_1 & + 2\omega \cdot a_{-1} + q_2 \cdot \epsilon = 0, \\ 2\omega \cdot b_1 - (\omega^2 + \frac{3}{4}) \cdot a_{-1} - \frac{3}{4} \sqrt{3}(1-2\mu) \cdot b_{-1} & + q_3 \cdot \epsilon = 0, \\ -2\omega \cdot a_1 - \frac{3}{4} \sqrt{3}(1-2\mu) \cdot a_{-1} & - (\omega^2 + \frac{9}{4}) \cdot b_{-1} + q_4 \cdot \epsilon = 0, \end{aligned} \right\} \quad (34)$$

where q_1 , q_2 , q_3 and q_4 are known functions of ϵ , which are small quantities of the second order.

From the first two equations (34) we find a_{-1} and b_{-1} in terms of a_1 and b_1 :

$$\begin{aligned} 2\omega \cdot a_{-1} &= \frac{3}{4} \sqrt{3}(1-2\mu) \cdot a_1 + (\omega^2 + \frac{9}{4}) \cdot b_1 - q_2 \cdot \epsilon, \\ 2\omega \cdot b_{-1} &= -(\omega^2 + \frac{3}{4}) \cdot a_1 - \frac{3}{4} \sqrt{3}(1-2\mu) \cdot b_1 + q_1 \cdot \epsilon. \end{aligned} \quad (35)$$

Inserting these values of a_{-1} and b_{-1} in the last two equations (34), we get the two equations :

$$\begin{aligned} [\omega^4 - \omega^2 + \frac{5}{16} - \frac{2}{16}(1-2\mu)^2] \cdot a_1 &= [(\omega^2 + \frac{9}{4})q_1 - 2\omega q_4 - \frac{3}{4}\sqrt{3}(1-2\mu)q_2] \cdot \epsilon, \\ [\omega^4 - \omega^2 + \frac{5}{16} - \frac{2}{16}(1-2\mu)^2] \cdot b_1 &= [(\omega^2 + \frac{3}{4})q_2 + 2\omega q_3 - \frac{3}{4}\sqrt{3}(1-2\mu)q_1] \cdot \epsilon. \end{aligned} \quad (36)$$

Introducing now in the equations (36) the values already chosen for the orbit-constant a_1 and the phase-constant b_1 :

$$a_1 = \epsilon, \quad b_1 = 0, \quad (37)$$

we get the two equations :

$$\omega^4 - \omega^2 + \frac{5}{16} - \frac{2}{16}(1-2\mu)^2 - (\omega^2 + \frac{9}{4})q_1 + \frac{3}{4}\sqrt{3}(1-2\mu)q_2 + 2\omega q_4 = 0, \quad (38)$$

$$(\omega^2 + \frac{3}{4})q_2 + 2\omega q_3 - \frac{3}{4}\sqrt{3}(1-2\mu)q_1 = 0, \quad (39)$$

where (38) corresponds to (22). In what follows we shall consider equation (38), reverting to equation (39) later on.

We now limit the analysis by only considering values of μ in the *neighbourhood* of the critical value μ_0 , the smaller root of the equation :

$$\mu_0(1 - \mu_0) = \frac{1}{27}. \quad (40)$$

The corresponding value of ω , determined with the aid of the variational equations, is found from (23) :

$$\omega_0^2 = \frac{1}{2}, \quad \omega_0 = \frac{1}{2}\sqrt{2}. \quad (41)$$

In the computations that follow we can exclude the negative value of ω_0 .

From (40) we find

$$\mu_0 = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{2}{27}}. \quad (42)$$

The difference between the great mass $1 - \mu_0$ and the small mass μ_0 is

$$1 - 2\mu_0 = \sqrt{\frac{2}{27}}. \quad (43)$$

For brevity we introduce the quantity k :

$$k = \sqrt{\frac{2}{27}}. \quad (44)$$

As we have assumed that μ lies in the neighbourhood of μ_0 , or that the mass-product lies in the neighbourhood of $\frac{1}{27}$, we can put

$$\mu(1 - \mu) = \frac{1}{27} + \delta, \quad (45)$$

where δ is infinitely small.

We now revert to equation (38). Introducing (45) in (38), we get

$$(\omega^2 - \frac{1}{2})^2 + \frac{2}{4}\delta - (\omega^2 + \frac{9}{4})q_1 + \frac{3}{4}\sqrt{3}(1-2\mu)q_2 + 2\omega q_4 = 0. \quad (46)$$

The last three terms in equation (46) are small quantities of the second order. As only terms of the second order are retained in this equation, we can substitute ω_0 and μ_0 for ω and μ in these terms. We therefore introduce in the last terms of (46)

$$\omega = \omega_0 = \frac{1}{2}\sqrt{2}, \quad 1 - 2\mu = 1 - 2\mu_0 = k, \quad (47)$$

and we get the new equation

$$(\omega^2 - \frac{1}{2})^2 + \frac{27}{4}\delta - \frac{11}{4}q_1 + \frac{3}{4}\sqrt{3}kq_2 + \sqrt{2}q_4 = 0. \tag{48}$$

The condition for real values of ω and therefore the condition for infinitesimal periodic orbits around L_4 is

$$27\delta \leq 11q_1 - 3\sqrt{3}kq_2 - 4\sqrt{2}q_4, \tag{49}$$

and we have thus found the upper limit of the mass-product

$$\frac{1}{27}[1 + 11q_1 - 3\sqrt{3}kq_2 - 4\sqrt{2}q_4]. \tag{50}$$

§ 6. We now turn to the computation of the quantities q_1, q_2, q_3 and q_4 , which are functions of μ and ϵ of the following form :—

$$q_n = f_n(\mu)\epsilon^2, \qquad (n = 1, 2, 3, 4). \tag{51}$$

As in this equation only terms of the second order are retained, we can in the computation of $f_n(\mu)$ substitute μ_0 and ω_0 for μ and ω . The computations are based on equations (33), which with $\mu = \mu_0$ and $\omega = \omega_0$ become

$$\begin{aligned} a_1 &= \epsilon, & a_{-1} &= \frac{3}{8}\sqrt{6}k\epsilon, \\ b_1 &= 0, & b_{-1} &= -\frac{5}{8}\sqrt{2}\epsilon. \end{aligned} \tag{52}$$

With the values given by (52) we compute, as already explained, the coefficients of the second order in the first column of the coefficient scheme corresponding to $\xi^2, \xi\eta$ and η^2 . The results are as follows :—

ξ^2	$\frac{5}{6}\frac{5}{4}\epsilon^2$
$\xi\eta$	$-\frac{1}{6}\frac{5}{4}\sqrt{3}k\epsilon^2$
η^2	$\frac{2}{6}\frac{5}{4}\epsilon^2$

(53)

Introducing these values in the differential equations (12), we get for the determination of a_0 and b_0 the two equations :

$$\begin{aligned} \frac{3}{4}a_0 + \frac{3}{4}\sqrt{3}kb_0 &= -\frac{7}{1}\frac{5}{2}\frac{5}{8}k\epsilon^2, \\ \frac{3}{4}\sqrt{3}ka_0 + \frac{9}{4}b_0 &= -\frac{8}{1}\frac{5}{9}\frac{5}{2}\sqrt{3}\epsilon^2, \end{aligned} \tag{54}$$

which have the following roots :—

$$a_0 = -\frac{1}{1}\frac{6}{2}\frac{5}{8}k\epsilon^2, \qquad b_0 = \frac{6}{3}\frac{5}{8}\frac{5}{4}\sqrt{3}\epsilon^2. \tag{55}$$

Next we compute the coefficients of the second order in the coefficient scheme corresponding to $\cos 2\omega t$ and $\sin 2\omega t$ and $\xi^2, \xi\eta$ and η^2 .

The results are shown below :

	$\cos 2\omega t$	$\sin 2\omega t$
ξ^2	$\frac{9}{6}\frac{5}{4}\epsilon^2$	$\frac{3}{8}\sqrt{6}k\epsilon^2$
$\xi\eta$	$\frac{1}{6}\frac{5}{4}\sqrt{3}k\epsilon^2$	$-\frac{5}{16}\sqrt{2}\epsilon^2$
η^2	$-\frac{2}{6}\frac{5}{4}\epsilon^2$	0

(56)

Introducing these values in the differential equations (12), we get for the determination of a_2 , a_{-2} , b_2 and b_{-2} the four equations :

$$\begin{aligned} \frac{1}{4}a_2 + \frac{3}{4}\sqrt{3}kb_2 + 2\sqrt{2}b_{-2} &= -\frac{9}{128}k\epsilon^2, \\ \frac{3}{4}\sqrt{3}ka_2 + \frac{1}{4}b_2 - 2\sqrt{2}a_{-2} &= \frac{1}{96}\sqrt{3}\epsilon^2, \\ -2\sqrt{2}b_2 + \frac{1}{4}a_{-2} + \frac{3}{4}\sqrt{3}kb_{-2} &= -\frac{1}{96}\sqrt{6}\epsilon^2, \\ 2\sqrt{2}a_2 + \frac{3}{4}\sqrt{3}ka_{-2} + \frac{1}{4}b_{-2} &= -\frac{6}{64}\sqrt{2}k\epsilon^2, \end{aligned} \quad (57)$$

which have the roots :

$$\begin{aligned} a_2 &= -\frac{1}{128}k\epsilon^2, & a_{-2} &= -\frac{1}{864}\sqrt{6}\epsilon^2, \\ b_2 &= \frac{7}{3456}\sqrt{3}\epsilon^2, & b_{-2} &= -\frac{1}{96}\sqrt{2}k\epsilon^2. \end{aligned} \quad (58)$$

With these values of a_0 , b_0 , a_2 , a_{-2} , b_2 and b_{-2} we can compute the coefficients of the third order in the coefficient scheme corresponding to $\cos \omega t$ and $\sin \omega t$ and ξ^2 , $\xi\eta$, η^2 , ξ^3 , $\xi^2\eta$, $\xi\eta^2$ and η^3 . The result is shown in the following scheme :—

	$\cos \omega t$	$\sin \omega t$
ξ^2	$-\frac{5}{1024}k\epsilon^3$	$-\frac{2}{27048}\sqrt{6}\epsilon^3$
$\xi\eta$	$\frac{1}{6012}\sqrt{3}\epsilon^3$	$\frac{4}{512}\sqrt{2}k\epsilon^3$
η^2	$\frac{6}{384}k\epsilon^3$	$-\frac{5}{27048}\sqrt{6}\epsilon^3$
ξ^3	$\frac{1}{128}\epsilon^3$	$\frac{4}{1024}\sqrt{6}k\epsilon^3$
$\xi^2\eta$	$-\frac{1}{64}\sqrt{3}k\epsilon^3$	$-\frac{5}{1024}\sqrt{2}\epsilon^3$
$\xi\eta^2$	$\frac{2}{128}\epsilon^3$	$\frac{2}{1024}\sqrt{6}k\epsilon^3$
η^3	0	$-\frac{3}{1024}\sqrt{2}\epsilon^3$

(59)

Introducing these values in the differential equations (12), we get for the more exact determination of a_1 , a_{-1} , b_1 and b_{-1} a system of four equations corresponding to (34). Comparing the two systems of equations we can determine the values of q_1 , q_2 , q_3 and q_4 :

$$\begin{aligned} q_1 &= \frac{4}{13824}\epsilon^2, & q_3 &= \frac{3}{1536}\sqrt{6}k\epsilon^2, \\ q_2 &= \frac{7}{4608}\sqrt{3}k\epsilon^2, & q_4 &= \frac{1}{4608}\sqrt{2}\epsilon^2. \end{aligned} \quad (60)$$

§ 7. With the values of q_1 , q_2 , q_3 and q_4 thus found we can show that equation (39) is satisfied. As only terms of the second order are retained in the equation, we can put

$$\mu = \mu_0, \quad \omega = \frac{1}{2}\sqrt{2}, \quad (61)$$

and the equation takes the form

$$5q_2 + 4\sqrt{2}q_3 - 3\sqrt{3}q_1 = 0. \quad (62)$$

Inserting the values of q_1 , q_2 and q_3 from (60), we find that (62) and consequently (39) is satisfied.

The second equation of condition, equation (38), is satisfied when $(\omega^2 - \frac{1}{2})^2$ has the value indicated by (48). Inserting the values of q_1 , q_3 and q_4 given in (60) we get the equation

$$(\omega^2 - \frac{1}{2})^2 + \frac{2}{4} \delta - \frac{2}{8} \frac{9}{6} \frac{5}{4} \epsilon^2 = 0. \quad (63)$$

The condition for real values of ω and hence for infinitesimal periodic orbits around L_4 is

$$27\delta \leq \frac{2}{2} \frac{9}{16} \epsilon^2, \quad (64)$$

and the upper limit of the mass product is

$$\frac{1}{27} [1 + \frac{2}{2} \frac{9}{16} \epsilon^2]. \quad (65)$$

It appears from (65) that periodic orbits around L_4 exist for values of the mass-product greater than $\frac{1}{27}$.

Now let us assume

$$\delta > 0. \quad (66)$$

We can then determine an orbit-constant ϵ_0 by the equation

$$\frac{2}{2} \frac{9}{16} \epsilon_0^2 = 27\delta, \quad (67)$$

which, inserting in (63), gives

$$(\omega^2 - \frac{1}{2})^2 - \frac{2}{8} \frac{9}{6} \frac{5}{4} (\epsilon^2 - \epsilon_0^2) = 0. \quad (68)$$

The condition for real values of ω is

$$\epsilon \geq \epsilon_0. \quad (69)$$

For $\epsilon = \epsilon_0$ there is only one value of $\omega^2 (\omega^2 = \frac{1}{2})$, corresponding to one periodic orbit (the limiting orbit). For $\epsilon > \epsilon_0$ there are two values of ω^2 , corresponding to two periodic orbits (short- and long-periodic). For $\epsilon < \epsilon_0$ there exist no periodic orbits.

If, however, we assume

$$\delta = 0, \quad (70)$$

we get from (63)

$$(\omega^2 - \frac{1}{2})^2 - \frac{2}{8} \frac{9}{6} \frac{5}{4} \epsilon^2 = 0. \quad (71)$$

For $\epsilon = 0$ there is only one value of $\omega^2 (\omega^2 = \frac{1}{2})$, corresponding to the libration point L_4 . For $\epsilon > 0$ there are two values of ω^2 , corresponding to two periodic orbits (short- and long-periodic).

Finally, let us assume

$$\delta < 0. \quad (72)$$

We can then determine ϵ_1 by the equation

$$\frac{2}{2} \frac{9}{16} \epsilon_1^2 = -27\delta, \quad (73)$$

which, inserting in (63), gives

$$(\omega^2 - \frac{1}{2})^2 - \frac{2}{8} \frac{9}{6} \frac{5}{4} (\epsilon^2 + \epsilon_1^2) = 0. \quad (74)$$

For $\epsilon = 0$ there are two values of ω^2 , corresponding to the libration point L_4 . For $\epsilon > 0$ there are two values of ω^2 , corresponding to two periodic orbits (short- and long-periodic).

§ 8. We shall now illustrate the previous results with the aid of a few figures.

Fig. 2 corresponds to equation (67), showing ϵ_0 as a function of δ . The curve is one branch of a parabola. A point to the left of the curve corresponds to two periodic orbits with the same orbit-constant, but with different periods. A point on the curve corresponds to one periodic orbit. For a point to the right of the curve there is no corresponding periodic orbit.

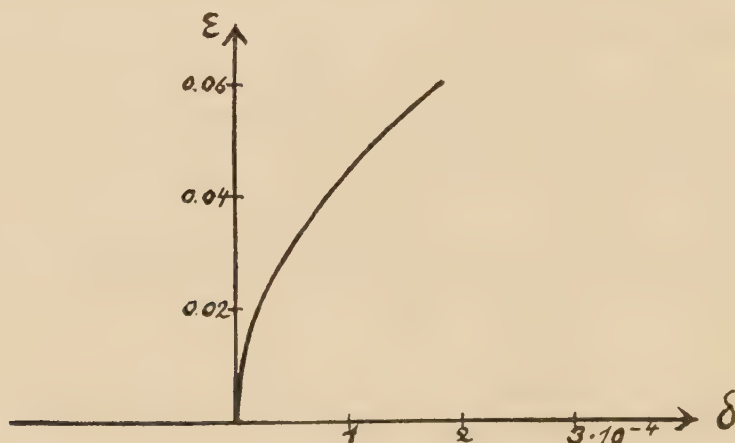


FIG. 2.—The Curve shows that Periodic Orbits around L_4 exist for Values of the Mass-product greater than $\frac{1}{27}$.

Fig. 3 corresponds to the equations (68), (71) and (74). The abscissa is ϵ , the ordinate is $\omega^2 - \frac{1}{2}$. Seven curves corresponding to different values of ϵ_0 and ϵ_1 are shown; the values of ϵ_0 and ϵ_1 are shown below:

No.	1	2	3	4	5	6	7
ϵ_0	...	...	...	0.00	0.02	0.04	0.06
ϵ_1	0.06	0.04	0.02	0.00	...	...	...
$10^6 \cdot \delta$	-182	-81	-20	0	+20	+81	+182

(75)

The curves are branches of hyperbolæ; the asymptotes form the curve corresponding to $\delta=0$, i.e. to the mass-product $\frac{1}{27}$. The curves that cut the ordinate axis (the ϵ_1 curves) correspond to mass-products smaller than $\frac{1}{27}$, while the curves that cut the abscissa axis (the ϵ_0 curves) correspond to mass-products *greater* than $\frac{1}{27}$. That part of the curves which lies above the abscissa axis corresponds to short-period orbits, while the part below the abscissa axis corresponds to long-period orbits. The run of the ϵ_0 curves shows that there is a continuous transition from short-period to long-period orbits. As the orbit-constant ϵ of the short-period orbit decreases the period increases, and the orbit tends to the *limiting orbit* with orbit-constant ϵ_0 and period $2\sqrt{2}\pi$. When the limiting orbit has been

reached both orbit-constant and period increase. The limiting orbit separates short-period and long-period orbits. Taken together these orbits and their continuations in both directions form a class.

The orbits of the class that corresponds to $\epsilon_0 = 0$ behave in a similar way, only the limiting orbit is here reduced to a point (the libration point).

Conditions are somewhat different regarding the ϵ_1 curves, as these form two separate branches. Starting with the branch above the abscissa

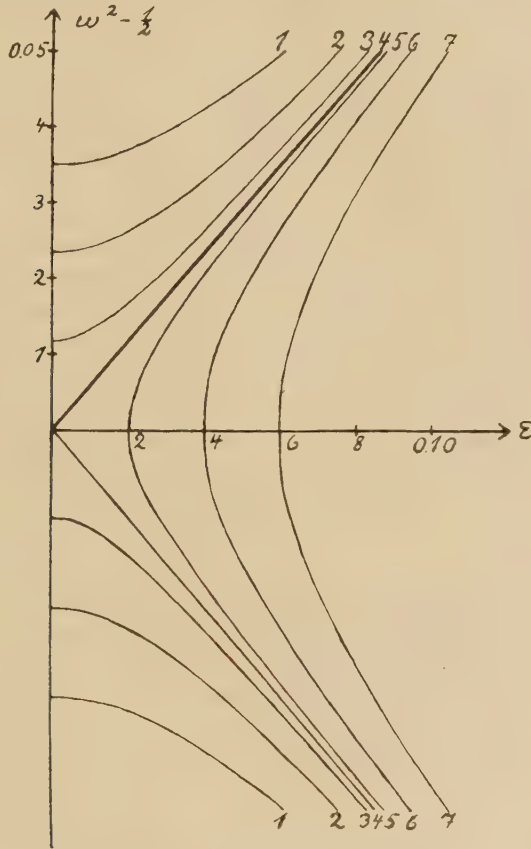


FIG. 3.—The Curves show the Dependence of Period ($\frac{2\pi}{\omega}$) on Size of Orbit (ϵ) for Periodic Orbits around L_4 for Values of the Mass-product in the Neighbourhood of $\frac{1}{27}$.

axis that corresponds to short-period orbits, we find as before that the period increases as the orbit-constant decreases. In the libration point the period reaches a relative maximum $2\sqrt{2\pi}(1 - \frac{1}{2}\sqrt{1770\epsilon_1})$. Continuing now along the other branch of the curve we start at the libration point where the period has the relative minimum $2\sqrt{2\pi}(1 + \frac{1}{2}\sqrt{1770\epsilon_1})$. Then follow orbits with increasing period as the orbit-constant increases. No orbits correspond to values of the period between $2\sqrt{2\pi}(1 - \frac{1}{2}\sqrt{1770\epsilon_1})$ and $2\sqrt{2\pi}(1 + \frac{1}{2}\sqrt{1770\epsilon_1})$. It is possible, but not certain, that in this case also the short- and long-period orbits belong to the same class.

§ 9. After this investigation of the equation of condition (63) we return to the problem of determining the Fourier coefficients of ξ and η .

The final determination of the coefficients of the first order a_{-1} and b_{-1} is carried out with the aid of equations (35), inserting $a_1 = \epsilon$ and $b_1 = 0$ and further inserting the values of q_1 and q_2 given by (60). In this way we get the following values:—

$$\begin{aligned} a_1 &= \epsilon, & 2\omega \cdot a_{-1} &= \frac{3}{4} \sqrt{3} (1 - 2\mu) \epsilon - \frac{7}{4} \frac{2}{5} \frac{0}{8} \frac{5}{8} \sqrt{3} k \epsilon^3, \\ b_1 &= 0, & 2\omega \cdot b_{-1} &= -(\omega^2 + \frac{3}{4}) \epsilon + \frac{4}{1} \frac{4}{3} \frac{7}{8} \frac{8}{2} \frac{5}{4} \epsilon^3. \end{aligned} \quad (76)$$

It should be emphasized that a_{-1} and b_{-1} on account of the presence of ω in equations (76) contain terms of the second order also. This circumstance is of importance in the determination of the coefficients of the second order a_0 , b_0 , a_2 , a_{-2} , b_2 and b_{-2} .

In the preliminary determination of these coefficients we computed the quantities of the second order in the coefficient scheme with the aid of the values of a_1 , a_{-1} , b_1 and b_{-1} given in (52). The results were given in the schemes (53) and (56). If, however, in computing these schemes we use the values of a_{-1} and b_{-1} given in (76), terms of the third order will appear in the coefficient scheme, and the expressions for the unknown coefficients a_0 , b_0 , a_2 , a_{-2} , b_2 and b_{-2} will therefore contain terms of the third order. As, however, the introduction of the expressions (76) in the coefficient scheme would give rather complicated expressions, we shall not carry out the computation of the terms of the third order in a_0 , b_0 , a_2 , a_{-2} , b_2 and b_{-2} .

The coefficients a_3 , a_{-3} , b_3 and b_{-3} , which are of the third order, are computed from the columns corresponding to $\cos 3\omega t$ and $\sin 3\omega t$ in the coefficient scheme. We first compute the coefficients of the third order in the coefficient scheme corresponding to $\cos 3\omega t$ and $\sin 3\omega t$ and ξ^2 , $\xi\eta$, η^2 , ξ^3 , $\xi^2\eta$, $\xi\eta^2$ and η^3 , inserting the values of a_1 , a_{-1} , b_1 and b_{-1} given in (52). The result is shown in the scheme below:

	$\cos 3\omega t$	$\sin 3\omega t$
ξ^2	$\frac{4}{1} \frac{7}{9} \frac{2}{2} k \epsilon^3$	$-\frac{4}{2} \frac{8}{7} \frac{2}{6} \frac{3}{4} \frac{8}{8} \sqrt{6} \epsilon^3$
$\xi\eta$	$-\frac{2}{6} \frac{5}{9} \frac{7}{1} \frac{2}{2} \sqrt{3} \epsilon^3$	$-\frac{4}{1} \frac{3}{5} \frac{3}{8} \sqrt{2} k \epsilon^3$
η^2	$-\frac{6}{3} \frac{5}{8} \frac{4}{4} k \epsilon^3$	$-\frac{3}{2} \frac{9}{7} \frac{5}{6} \frac{4}{8} \sqrt{6} \epsilon^3$
ξ^3	$-\frac{3}{1} \frac{7}{2} \frac{8}{8} \epsilon^3$	$\frac{2}{1} \frac{1}{0} \frac{9}{2} \frac{4}{4} \sqrt{6} k \epsilon^3$
$\xi^2\eta$	$\frac{1}{6} \frac{5}{4} \sqrt{3} k \epsilon^3$	$-\frac{4}{1} \frac{5}{0} \frac{5}{2} \frac{4}{4} \sqrt{2} \epsilon^3$
$\xi\eta^2$	$-\frac{2}{1} \frac{5}{2} \frac{8}{8} \epsilon^3$	$-\frac{7}{1} \frac{5}{0} \frac{5}{2} \frac{4}{4} \sqrt{6} k \epsilon^3$
η^3	0	$\frac{1}{1} \frac{2}{0} \frac{5}{2} \frac{4}{4} \sqrt{2} \epsilon^3$

(77)

Introducing these values in the differential equations (12), we get four equations for the determination of a_3 , a_{-3} , b_3 and b_{-3} :

$$\begin{aligned}
& \frac{21}{4}a_3 + \frac{3}{4}\sqrt{3}kb_3 + 3\sqrt{2}b_{-3} = \frac{16691}{13824}\epsilon^3, \\
& \frac{3}{4}\sqrt{3}ka_3 + \frac{27}{4}b_3 - 3\sqrt{2}a_{-3} = -\frac{2329}{4808}\sqrt{3}k\epsilon^3, \\
& -3\sqrt{2}b_3 + \frac{21}{4}a_{-3} + \frac{3}{4}\sqrt{3}kb_{-3} = \frac{1021}{2304}\sqrt{6}k\epsilon^3, \\
& 3\sqrt{2}a_3 + \frac{3}{4}\sqrt{3}ka_{-3} + \frac{27}{4}b_{-3} = \frac{7973}{6912}\sqrt{2}\epsilon^3,
\end{aligned} \tag{78}$$

which have the roots :

$$\begin{aligned}
a_3 &= \frac{7595}{55296}\epsilon^3, & a_{-3} &= \frac{1405}{36864}\sqrt{6}k\epsilon^3, \\
b_3 &= -\frac{1037}{18432}\sqrt{3}k\epsilon^3, & b_{-3} &= \frac{10951}{110592}\sqrt{2}\epsilon^3.
\end{aligned} \tag{79}$$

§ 10. Owing to the complication of the expressions we did not carry out the computation of the terms of the third order in the coefficients of second order $a_0, b_0, a_2, a_{-2}, b_2$ and b_{-2} .

In the case, however, of the *limiting orbits* the terms of the second order in a_{-1} and b_{-1} disappear, and consequently the terms of the third order in $a_0, b_0, a_2, a_{-2}, b_2$ and b_{-2} will also disappear. We therefore carry out the determination of the Fourier coefficients of ξ and η for these limiting orbits.

For the limiting orbits the equation of condition (63) yields only one value of ω^2 ($\omega^2 = \frac{1}{2}$). The orbit-constant of the limiting orbit is given by equation (67) :

$$\frac{295}{216}\epsilon_0^2 = 27\delta, \tag{80}$$

while δ is given by equation (45) :

$$\mu(1 - \mu) = \frac{1}{27} + \delta. \tag{81}$$

From (80) and (81) we obtain an equation which shows the connection between ϵ_0 and μ :

$$27\mu(1 - \mu) = 1 + \frac{295}{216}\epsilon_0^2, \tag{82}$$

and from this we find, retaining terms up to the second order only,

$$1 - 2\mu = k[1 - \frac{295}{2434}\epsilon_0^2]. \tag{83}$$

Introducing now $\omega = \frac{1}{2}\sqrt{2}$ and the value of $1 - 2\mu$ given by (83) in equations (76) we get the following values of the coefficients of the first order for the limiting orbits :—

$$\begin{aligned}
a_1 &= \epsilon, & a_{-1} &= \frac{3}{8}\sqrt{6}k[\epsilon_0 - \frac{58385}{28496}\epsilon_0^3], \\
b_1 &= 0, & b_{-1} &= -\frac{5}{8}\sqrt{2}[\epsilon_0 - \frac{8957}{3456}\epsilon_0^3].
\end{aligned} \tag{84}$$

It appears from (84) that a_{-1} and b_{-1} contain no terms of the second order, and it is therefore possible to find the coefficients of the second order by substituting ϵ_0 for ϵ in (55) and (58) :

$$\begin{aligned}
a_0 &= -\frac{165}{128}k\epsilon_0^2, & b_0 &= \frac{65}{384}\sqrt{3}\epsilon_0^2, \\
a_2 &= -\frac{113}{128}k\epsilon_0^2, & a_{-2} &= -\frac{127}{864}\sqrt{6}\epsilon_0^2, \\
b_2 &= \frac{79}{3456}\sqrt{3}\epsilon_0^2, & b_{-2} &= -\frac{13}{96}\sqrt{2}k\epsilon_0^2.
\end{aligned} \tag{85}$$

In a similar way the coefficients of the third order are found from (79) :

$$\begin{aligned}
a_3 &= \frac{7595}{55296}\epsilon_0^3, & a_{-3} &= \frac{1405}{36864}\sqrt{6}k\epsilon_0^3, \\
b_3 &= -\frac{1037}{18432}\sqrt{3}k\epsilon_0^3, & b_{-3} &= \frac{10951}{110592}\sqrt{2}\epsilon_0^3.
\end{aligned} \tag{86}$$

Introducing now (84), (85) and (86) in (14) we get the Fourier expressions for the limiting orbits :

$$\left. \begin{aligned} \xi &= -\frac{1}{128}k\epsilon_0^2 + \epsilon_0 \cos \omega_0 t + \frac{3}{8}\sqrt{6}k[\epsilon_0 - \frac{5}{8}\frac{3}{8}\frac{5}{8}\epsilon_0^3] \sin \omega_0 t \\ &\quad - \frac{1}{128}k\epsilon_0^2 \cos 2\omega_0 t - \frac{1}{8}\frac{7}{64}\sqrt{6}\epsilon_0^2 \sin 2\omega_0 t \\ &\quad + \frac{7}{55}\frac{9}{2}\frac{5}{8}\epsilon_0^3 \cos 3\omega_0 t + \frac{1}{3}\frac{4}{6}\frac{5}{8}\frac{5}{4}\sqrt{6}k\epsilon_0^3 \sin 3\omega_0 t, \\ \eta &= \frac{6}{384}\sqrt{3}\epsilon_0^2 - \frac{5}{8}\sqrt{2}[\epsilon_0 - \frac{8}{3}\frac{5}{8}\frac{7}{6}\epsilon_0^3] \sin \omega_0 t \\ &\quad + \frac{7}{34}\frac{9}{5}\frac{5}{6}\sqrt{3}\epsilon_0^2 \cos 2\omega_0 t - \frac{1}{9}\frac{3}{6}\sqrt{2}k\epsilon_0^2 \sin 2\omega_0 t \\ &\quad - \frac{1}{18}\frac{3}{4}\frac{7}{32}\sqrt{3}k\epsilon_0^3 \cos 3\omega_0 t + \frac{1}{11}\frac{0}{6}\frac{5}{8}\frac{1}{2}\sqrt{2}\epsilon_0^3 \sin 3\omega_0 t, \end{aligned} \right\} \quad (87)$$

where $\omega_0 = \frac{1}{2}\sqrt{2}$.

The Jacobian constant for the limiting orbits is computed with the aid

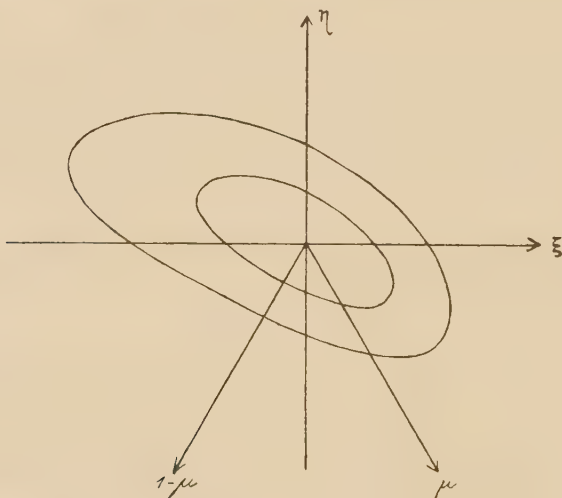


FIG. 4.—The Limiting Orbits for the Periodic Orbits around L_4 , when the Small Mass has the Values $\mu_1 = \mu_0 + 0.00055$ (Inner Orbit) and $\mu_2 = \mu_0 + 0.00220$ (Outer Orbit).

of column 1 of the coefficient scheme. The limiting orbits, all having the same period, have also the same Jacobian constant, viz.

$$C = 2. \quad (88)$$

§ 11. In order to illustrate equations (87) we introduce the values 0.1 and 0.2 for ϵ_0 . With $\epsilon_0 = 0.1$ we get from (87) the Fourier expressions :

$$\begin{aligned} 10^5 \cdot \xi &= -1190 + 10000 \cos \omega_0 t + 8291 \sin \omega_0 t \\ &\quad - 79 \cos 2\omega_0 t - 360 \sin 2\omega_0 t \\ &\quad + 14 \cos 3\omega_0 t + 9 \sin 3\omega_0 t, \\ 10^5 \cdot \eta &= 293 - 8610 \sin \omega_0 t \\ &\quad + 40 \cos 2\omega_0 t - 177 \sin 2\omega_0 t \\ &\quad - 9 \cos 3\omega_0 t + 14 \sin 3\omega_0 t, \end{aligned} \quad (89)$$

and with $\epsilon_0 = 0.2$ we get

$$\begin{aligned}
 10^5 \cdot \xi &= -4759 + 20000 \cos \omega_0 t + 15461 \sin \omega_0 t \\
 &\quad - 317 \cos 2\omega_0 t - 1440 \sin 2\omega_0 t \\
 &\quad + 110 \cos 3\omega_0 t + 69 \sin 3\omega_0 t, \\
 10^5 \cdot \eta &= 1173 \quad - 15845 \sin \omega_0 t \\
 &\quad + 158 \cos 2\omega_0 t - 707 \sin 2\omega_0 t \\
 &\quad - 72 \cos 3\omega_0 t + 112 \sin 3\omega_0 t.
 \end{aligned} \tag{90}$$

The corresponding values of μ are found with the aid of equation (82) or (83). Denoting the values of μ corresponding to $\epsilon_0 = 0.1$ and $\epsilon_0 = 0.2$ by μ_1 and μ_2 , respectively we get from (82):

$$\begin{aligned}
 \mu_1 &= 0.03907 = \mu_0 + 0.00055, \\
 \mu_2 &= 0.04072 = \mu_0 + 0.00220.
 \end{aligned} \tag{91}$$

Fig. 4 shows the form of the two limiting orbits.

Viborg, Denmark:
1933 August.

STELLAR PARALLAXES DETERMINED PHOTOGRAPHICALLY AT THE CAPE OBSERVATORY. (FIFTH LIST.)

(Communicated by H.M. Astronomer)

The present list of parallaxes has been derived from photographs taken with the 24-inch Victoria telescope at the Cape Observatory (scale 1 mm. = 30"). The stars are numbered in rotation in continuation of the previous lists, which were published in *M.N.*, 89, 497, 1929; 90, 190, 1929; 91, 822, 1931; and 92, 730, 1932.

For most stars the Durchmusterung number is given as in the Henry Draper Catalogue, viz. B.D. number down to Dec. -23° , the Co.D. number south of this limit to -52° , and the C.P.D. number south of -52° . For a few stars south of -52° contained in Co.D. but not in the C.P.D. the number in the former is given with an indication that the Co.D. number is given. For a few stars too faint to appear in the Durchmusterungs the necessary references for identification are given in the source from which the star was selected for parallax determination.

The magnitudes and spectral types, for stars contained in the Draper Catalogue, have been extracted from that catalogue. For the fainter stars they have been extracted from other sources or have been estimated from the plates.

Two values of the proper-motion in R.A. are given. The first (as also the proper-motion in declination) is based on meridian observations or from comparison of photographs taken at fairly distant epochs; the second value is the relative proper-motion derived from the parallax plates themselves. The stars on the observing programme have generally been selected on

account of large proper-motion, but it will be seen from the tabular data that several of these proper-motions were spurious. In the last column the first value of the probable error refers to the parallax, and the second value to the proper-motion in R.A. These probable errors are based on the probable errors for a plate of unit weight given in the eighth column.

The parallaxes are relative to the mean of four or five comparison stars, which are generally of magnitude 11.0 to 11.5.

This list contains the well-known stars Kapteyn's proper-motion star and the brightest component of the triple star α^2 (=40) Eridani.

No.	Name and D.M. No.	Mag. Vis. Phot.	Type	R.A. Dec. (1900)	Proper-Motion		P.E. Unit Weight.	Parallax. Unit ".001	P.E. Unit ".001
					R.A. Unit	Dec. ".001			
417	-58° 98	9.9 10.7	G5	^{h m s} 1 18 0 -58° 30'.0	+ 337 + 330 + 259		25	+ 11	± 8 ± 7
418	-26° 957	9.1 9.2	G0	2 34 43 -26° 44'.6	+ 224 - 272 + 152		20	+ 44	± 7 ± 6
419	-51° 634	9.3 8.7	K0	2 37 44 -51° 52'.3	+ 296 + 170 + 322		22	+ 29	± 9 ± 9
420	Wolf 1132	... 11	...	2 38 56 -9° 14'.0	+ 977 - 688 + 635		21	+ 34	± 8 ± 8
421	α Fornacis -29° 1177	3.95 4.45	F8	3 7 49 -29° 22'.9	+ 327 + 648 + 325		20	+ 54	± 7 ± 6
422	82 Eridani -43° 1028	4.30 5.08	G5	3 15 56 -43° 27'.1	+3075 + 748 +3059		22	+151	± 8 ± 8
423	-15° 595	8.0 8.8	G5	3 20 37 -15° 23'.0	- 189 - 297 - 141		16	- 22	± 6 ± 5
424	α^2 Eridani -7° 780	4.48 5.26	G5	4 10 40 -7° 48'.5	-2205 - 3435 -2274		22	+191	± 7 ± 7
425	ϵ Leporis -22° 1000	3.29 4.47	K5	5 1 14 -22° 30'.3	+ 26 - 67 + 6		26	+ 38	± 9 ± 11
426	-45° 1841 (Kapteyn's star)	9.2 10.6	K2	5 7 42 -44° 58'.8	+6599 - 5740 +6635		29	+248	± 10 ± 11
427	-75° 314	7.9 8.5	G0	5 28 56 -75° 45'.8	+ 148 - 300 - 93		21	+ 39	± 7 ± 8
428	γ Leporis -22° 1211	3.80 4.30	F8	5 40 18 -22° 28'.9	- 279 - 376 - 299		25	+134	± 8 ± 8
429	A.J. 852-3, 55 Tacubaya 7 ^h 8 ^m -15° 296	... 10.8	...	7 6 14 -14° 16'.3	- 436 + 438 - 479		23	+ 34	± 8 ± 4

No.	Name and D.M. No.	Mag. Vis. Phot.	Type	R.A. Dec. (1900)	Proper-Motion		P.E. Unit Weight. Unit ".001	Parallax. Unit ".001	P.E. Unit ".001
					R.A. Unit	Dec. ".001			
430	Tacubaya 7 ^h 8 ^m - 15° 307	... 11.2	...	^h ^m ^s 7 6 17 - 14° 15'.4	...	...	24	+ 22	± 8 ± 4
431	ρ Puppis - 23° 6828	2.88 3.30	F5	8 3 17 - 24° 1'.0	- 89 + 45 - 88		26	+ 51	± 9 ± 9
432	- 29° 6145	7.72 8.6	G5	8 21 43 - 29° 35'.7	+ 168 - 338 + 150		24	+ 51	± 8 ± 9
433	- 27° 5534	9.6 8.6	B9	8 25 1 - 27° 26'.5	+ 269 - 179 - 12		30	- 40	± 11 ± 13
434	- 27° 5536	... 8.8	...	8 25 2 - 27° 27'.4	...	...	26	- 21	± 9 ± 11
435	- 53° 1708	... 9.2	...	8 27 3 - 53° 56'.7	- 247 + 240 - 403		24	+ 11	± 8 ± 9
436	- 42° 4577	7.6 7.5	G5	8 41 10 - 42° 15'.8	0 - 310 0		24	+ 44	± 12 ± 10
437	12 Hydræ - 13° 2673	4.44 5.22	G5	8 41 39 - 13° 10'.9	+ 25 - 19 + 42		21	+ 8	± 8 ± 8
438	- 23° 7884	8.8 9.3	K5	8 50 36 - 24° 1'.0	- 358 + 164 - 357		22	+ 63	± 8 ± 8
439	U.O.C. 53	... 14	...	9 15 5 - 66° 46'.2	- 148 + 293 - 161		20	- 12	± 8 ± 5
440	Boss 2521 - 61° 1242	4.86 6.6	K0	9 18 33 - 61° 58'.7	- 7 - 15 - 1		23	+ 19	± 8 ± 5
441	- 59° 1362	... 9.6	...	9 19 13 - 59° 51'.5	- 829 + 140 - 824		23	+ 118	± 10 ± 11
442	- 47° 4996	8.8 8.7	G5	9 26 57 - 47° 10'.0	- 228 + 57 - 214		26	+ 27	± 10 ± 6
443	- 13° 2948	... 9.2	...	9 41 40 - 14° 4'.4	- 233 + 200 - 271		21	+ 18	± 8 ± 5
444	- 11° 2741	... 9.3	Mo	9 46 10 - 11° 49'.3	+ 1176 - 1425 + 1178		28	+ 37	± 11 ± 8
445	- 28° 7907	8.2 9.1	K2	10 2 35 - 28° 22'.8	- 404 0 - 6		22	+ 8	± 8 ± 4
446	- 36° 6180	7.7 9.2	G0	10 5 31 - 36° 16'.4	- 267 + 375 - 345		23	+ 43	± 9 ± 5

No.	Name and D.M. No.	Mag. Vis. Phot.	Type	R.A. Dec. (1900)	Proper-Motion		P.E. Unit Weight. Unit ".001	Parallax. Unit ".001	P.E. Unit ".001
					R.A. Unit	Dec. ".001			
447	-28° 8270	9.6 11.1	Ko	^{h m s} 10 29 21 -29° 1'.4	-306 + 127 -376		19	+ 30	± 7 ± 5
448	-59° 2507	... 8.9	...	10 39 4 -59° 31'.7	-266 + 160 -323		23	+ 13	± 8 ± 5
449	-40° 6271	... 9.3	...	10 40 40 -40° 45'.3	- 91 + 350 - 11		21	- 40	± 7 ± 5
450	-27° 7881	... 9.0	...	11 3 20 -27° 43'.6	-490 - 121 -500		22	+ 21	± 8 ± 5
451	-22° 3101	8.7 9.1	Go	11 9 52 -23° 6'.2	+306 + 43 +311		26	+ 25	± 12 ± 7
452	-56° 4554	... 8.7	...	11 26 3 -56° 35'.0	-595 + 60 -550		20	+ 51	± 7 ± 6
453	N.p. No. 454	... 10.0	...	11 40 32 -55° 7'.8	... + 17	...	26	- 14	± 10 ± 7
454	Co.D. -55° 4223	... 10.2	...	11 40 38 -55° 8'.6	-334 + 16	0	24	- 5	± 9 ± 6
455	-29° 9337	6.56 7.5	Go	11 42 16 -29° 43'.5	-285 - 248 -259		23	+ 60	± 8 ± 6
456	88 G Centauri -41° 6938	5.28 6.0	Fo	11 58 29 -41° 52'.4	+319 - 125 +355		24	+ 23	± 9 ± 8
457	-45° 7581	8.2 9.2	K2	12 4 47 -45° 39'.0	-378 - 60 -368		24	+ 46	± 9 ± 9
458	-30° 9942	9.49 9.4	K2	12 24 0 -30° 16'.7	+155 - 297 +296		28	+ 40	± 8 ± 10
459	-57° 5617	9.4 10.2	G5	12 34 0 -57° 21'.3	-413 -405	0	26	- 28	± 8 ± 11
460	-17° 3723	8.3 8.8	F8	12 47 56 -17° 57'.4	+313 - 819 +387		37	+ 11	± 14 ± 16
461	185 G Centauri -37° 8437	4.89 6.5	G5	13 6 28 -37° 16'.4	-390 + 30 -335		19	+ 18	± 6 ± 7
462	57 Virginis -19° 3653	5.32 6.8	Ko	13 10 34 -19° 24'.7	+296 - 128 +328		27	+ 46	± 9 ± 7
463	-22° 3557	... 9.4	...	13 13 48 -22° 29'.4	-335 - 66 -310		27	+ 3	± 9 ± 10

No.	Name and D.M. No.	Mag. Vis. Phot.	Type	R.A. Dec. (1900)	Proper-Motion		P.E. Unit Weight. Unit ".001	Parallax Unit ".001	P.E. Unit ".001
					R.A. Unit	Dec. ".001			
464	-27° 9298	9.6 9.7	Go	^{h m s} 13 27 52 -27° 9'.8	- 507 + 90 - 51		25 - 15	± 9 ± 9	
465	-67° 2325	6.84 7.6	Go	13 31 57 -67° 9'.5	+ 291 - 280 + 335		18 + 25	± 8 ± 6	
466	<i>i</i> Centauri -32° 9603	4.36 4.78	F5	13 40 0 -32° 32'.3	- 465 - 153 - 439		23 + 36	± 7 ± 8	
467	-50° 8092	7.54 7.9	G5	13 46 20 -50° 25'.6	- 646 - 27 - 606		21 + 55	± 7 ± 8	
468	<i>θ</i> Centauri -35° 9260	2.26 3.26	Ko	14 0 48 -35° 52'.7	- 530 - 528 - 525		24 + 42	± 8 ± 8	
469	-15° 4041	9.9 10.7	G5	15 4 44 -15° 59'.0	-1002 - 3545 -1016		23 + 40	± 8 ± 10	
470	-15° 4042	9.4 10.0	Go	15 4 45 -15° 54'.0	- 991 - 3544 - 971		21 + 60	± 8 ± 8	
471	-47° 9926	9.0 7.7	Go	15 15 40 -47° 33'.2	- 395 - 260 - 375		24 + 42	± 7 ± 8	
472	-57° 7690	7.3 8.2	G5	16 5 46 -57° 16'.5	- 843 - 1320 - 866		20 + 75	± 6 ± 7	
473	S.p No. 474	... 11	...	16 19 58 -60° 13'.9	... + 24	...	25 + 2	± 8 ± 9	
474	U.O.C. 48	... 11.5	...	16 20 3 -60° 13'.6	+ 916 + 628 + 31		22 - 6	± 7 ± 8	
475	-8° 4352	9.2 10.4	K5p	16 50 8 -8° 9'.1	- 858 - 887 - 847		25 + 150	± 8 ± 8	
476	N.p. No. 477	... 11	...	17 38 5 -57° 13'.8	... - 2	...	25 - 4	± 8 ± 10	
477	U.O.C. 48 3'.8 N.f. -57° 8716	... 11	...	17 38 15 -57° 13'.9	-1063 - 1418 -1162		25 + 146	± 8 ± 10	
478	-59° 7243	8.0 8.4	F8	18 7 36 -59° 25'.7	... - 270	...	20 + 8	± 6 ± 9	
479	Boss 4632 -18° 4926	Var. 5.8-6.6	F5p	18 15 30 -18° 54'.3	... + 1	...	24 - 24	± 7 ± 8	
480	-58° 7399	9.6 10.8	...	18 21 28 -58° 19'.3	... + 7	...	20 + 10	± 6 ± 7	
481	-58° 7400	9.6 10.8	...	18 21 31 -58° 19'.5	0 - 389 - 38		26 + 67	± 7 ± 9	
482	Boss 4756 -4° 4582	4.47 5.03	Go	18 41 52 -4° 51'.3	- 10 - 23 - 15		30 + 29	± 11 ± 13	

No.	Name and D.M. No.	Mag. Vis. Phot.	Type	R.A. Dec. (1900)	Proper-Motion		P.E. Unit Weight. Unit ".001	Parallax Unit ".001	P.E. Unit ".001
					R.A. Unit	Dec. ".001			
483	-15° 5243	9.8 10.4	Go	^h 19 ^m 2 ^s 16 -15° 23'.0	- 104 - 294 - 163		24	+ 35	± 7 ± 11
484	-47° 12773	9.7 10.8	K5	19 3 27 -47° 18'.0	- 61 - 594 - 39		23	+ 25	± 7 ± 9
485	-16° 5195	9.3 10.3	Ko	19 6 22 -15° 58'.7	+ 534 - 650 - 6		24	- 16	± 7 ± 9
486	ρ Sagittarii -18° 5322	3.95 4.09	A5	19 15 52 -18° 2'.1	- 24 + 20 - 46		31	+ 40	± 9 ± 12
487	-10° 5130	8.5 9.5	Ko	19 31 17 -10° 39'.5	- 287 - 265 - 270		22	+ 50	± 6 ± 7
488	-57° 9526	... 10.0	...	19 46 4 -57° 29'.0	0 - 390 - 55		26	+ 20	± 8 ± 10
489	-57° 9528	... 10.2	...	19 46 5 -57° 30'.6	... - 27	...	24	+ 14	± 7 ± 9
490	-57° 9530	... 10.0	...	19 46 7 -57° 31'.0	... - 14	...	34	- 11	± 10 ± 13
491	-16° 5509	8.2 8.8	Go	20 2 53 -16° 0'.5	+ 169 - 392 + 173		24	+ 8	± 9 ± 8
492	-15° 5584	8.6 9.2	Go	20 7 46 -15° 43'.5	- 345 - 255 - 339		26	+ 41	± 8 ± 11
493	-19° 5889	8.2 8.6	Go	20 35 6 -19° 8'.1	+ 51 - 437 + 43		28	+ 11	± 8 ± 10
494	-20° 6185	9.2 10.5	K5p	21 14 36 -20° 15'.3	- 182 - 742 - 83		26	+ 30	± 10 ± 12
495	-49° 13515	8.6 10.1	Ma	21 26 53 -49° 25'.8	0 - 800 - 13		23	+ 205	± 8 ± 10
496	42 Capricorni -14° 6102	5.28 6.06	G5	21 36 7 -14° 29'.6	- 121 - 308 - 122		17	+ 22	± 6 ± 7
497	-77° 1522	7.8 8.9	K2	21 44 8 -77° 48'.2	+ 285 - 220 + 237		19	+ 32	± 7 ± 8
498	τ Piscis Aust -33° 15941	5.06 5.9	F8	22 4 17 -33° 2'.4	+ 446 + 6 + 457		25	+ 27	± 8 ± 10
499	ν Indi -72° 2690	5.42 6.1	Go	22 16 2 -72° 44'.5	+ 1275 - 694 + 1304		26	+ 11	± 9 ± 10
500	-36° 15445	7.8 8.1	Go	22 27 44 -35° 57'.4	+ 385 - 143 + 318		19	+ 35	± 7 ± 8

FAINT DOUBLE STARS WITH COMMON PROPER MOTION IN THE SOUTHERN SKY. (SECOND LIST.)

W. J. Luyten.

In continuation of the list given a year ago (*M.N.*, 93, 1933, 196-208) data are given below for 91 more wide, faint pairs with common proper motion, found during the proper motion survey of the southern hemisphere. The arrangement of the table is identical to that given before, and it need only be added that the values for $\log P$ and w given in the last two columns have been calculated from formulæ (21) and (22) from the previous paper, and represent, respectively, the period of revolution and the expected annual orbital motion.

$\wedge$	R.A.		Dec.	m_1	m_2	q	s	μ	ψ	$\log P$	w
	^h	^m	[°] [']				["]	["]	[°]		["]
103	0	08.2	-52 52	12.5	14.7	<i>sf</i>	270	0.141	111	6.47	0.0004
104		19.5	-52 59	12.8	14.4	<i>nf</i>	141	0.053	93	6.49	02
105		36.0	-50 08	9.0	14.0	<i>nf</i>	4	0.250	109	3.13	138
106	1	16.0	-50 40	15.3	15.7	<i>np</i>	6	0.2	102	4.07	24
107		19.1	-55 15	11.6	13.2	<i>nf</i>	235	0.154	31	6.24	06
108		42.9	-52 23	12.2	13.1	<i>p</i>	11	0.119	90	4.46	18
109	2	01.8	-51 07	13.8	14.8	<i>np</i>	8	0.102	60	4.41	14
110	8	11.5	-49 57	13.3	13.6	<i>f</i>	87	0.103	318	5.90	05
111		12.0	-53 48	13.0	14.5	<i>s</i>	22	0.041	356	5.41	04
112		42.9	-49 46	13.8	14.7	<i>p</i>	4	0.086	236	4.04	17
113		49.0	-54 35	10.5	13.5	<i>n</i>	37	0.110	326	5.28	11
114		53.3	-54 29	12.9	13.5	<i>np</i>	5	0.173	309	3.78	39
115	10	22.9	-52 25	11.3	13.3	<i>sf</i>	159	0.078	272	6.25	04
116	11	18.0	-51 02	11.2	11.7	<i>nf</i>	31	0.115	311	5.06	13
117		19.0	-54 30	12.0	14.5	<i>p</i>	17	0.152	101	4.58	21
118		38.7	-52 50	13.7	14.0	<i>f</i>	13	0.033	307	5.19	04
119		39.4	-52 59	11.4	13.7	<i>sp</i>	21	0.115	257	4.78	16
120	12	00.1	-53 34	13.2	13.6	<i>np</i>	27	0.101	302	5.13	09
121		09.5	-51 13	9.7	12.8	<i>sf</i>	25	0.086	270	4.83	17
122		15.5	-53 00	9.2	9.5	<i>sp</i>	30	0.079	346	4.91	17
123		26.5	-49 59	12.5	12.6	<i>nf</i>	11	0.122	182	4.43	21
124		41.5	-51 37	14.7	15.1	<i>np</i>	16	0.095	290	4.97	08
125		46.2	-50 57	11.0	13.7	<i>sp</i>	53	0.155	303	5.22	15
126		53.9	-51 50	11.2	11.7	<i>np</i>	272	0.103	228	6.44	05
127	0	10.4	-32 52	12.4	13.8	<i>sf</i>	13	0.141	151	4.46	21
128	1	53.5	-36 44	12.4	12.7	<i>nf</i>	45	0.143	155	5.23	12
129	2	12.2	-31 28	12.7	13.2	<i>np</i>	105	0.711	68	5.16	34
130	3	08.6	-35 11	10.7	12.1	<i>np</i>	54	0.202	189	5.06	22
131		08.7	-30 25	12.7	13.8	<i>sp</i>	13	0.159	80	4.44	22
132		20.2	-36 44	14.5	14.7	<i>sf</i>	56	0.071	123	5.88	03
133	5	32.9	-33 56	12.7	14.1	<i>sp</i>	15	0.242	173	4.35	32
134	6	16.7	-31 06	12.5	12.6	<i>np</i>	30	0.090	325	5.17	10
135		30.5	-36 28	10.8	13.0	<i>sf</i>	22	0.148	146	4.64	23
136		33.1	-32 49	12.5	15.5	<i>sf</i>	11	0.112	177	4.48	17
137		37.2	-39 06	12.8	13.2	<i>sf</i>	30	0.090	356	5.21	09
138	7	23.3	-34 07	7.5	7.5	<i>nf</i>	17	0.182	295	3.99	81
139		54.3	-30 47	11.7	13.5	<i>nf</i>	11	0.124	312	4.35	23

Λ	R.A.	Dec.	m_1	m_2	q	s	μ	ψ	$\log P$	w
	^h ^m	[°] [']				"	"	[°]		"
140	7 59.9	-31 17	10.5	11.3	<i>sp</i>	47	0.155	319	5.08	0.0018
141	8 06.9	-33 24	12.1	14.3	<i>nf</i>	53	.122	288	5.43	09
142	27.7	-36 21	8.8	11.3	<i>sp</i>	47	.103	127	5.09	18
143	45.3	-32 01	11.7	14.3	<i>nf</i>	31	.383	188	4.53	43
144	59.1	-37 12	12.8	14.0	<i>np</i>	13	.089	288	4.70	12
145	9 07.5	-39 21	13.5	14.1	<i>nf</i>	15	.090	312	4.84	10
146	10.8	-29 54	14.1	14.6	<i>nf</i>	3	.288	294	3.35	63
147	43.2	-36 33	11.9	14.7	<i>f</i>	39	.089	114	5.35	08
148	10 25.1	-37 51	15.0	15.5	<i>sp</i>	39	.060	321	5.79	03
149	27.6	-39 36	12.5	15.6	<i>sf</i>	17	.093	326	4.85	11
150	36.0	-32 26	12.5	14.9	<i>p</i>	8	.094	271	4.35	17
151	37.4	-36 07	13.0	13.8	<i>np</i>	17	.078	303	4.94	09
152	11 38.2	-38 53	9.0	9.5	<i>np</i>	25	.163	291	4.47	40
153	41.8	-31 15	13.3	14.5	<i>np</i>	103	.064	283	6.24	03
154	48.7	-33 15	13.3	14.8	<i>sp</i>	10	.042	267	4.92	06
155	50.7	-37 42	13.5	15.8	<i>np</i>	8	.714	117	3.57	100
156	51.4	-31 38	8.0	8.0	<i>f</i>	22	.173	264	4.24	59
157	59.3	-32 43	12.7	13.2	<i>n</i>	15	.107	90	4.69	14
158	12 16.6	-33 31	12.7	12.9	<i>nf</i>	57	.061	267	{AB AC	
				13.5	<i>nf</i>	256				
159	17.4	-30 30	14.0	14.4	<i>nf</i>	10	.090	225	4.62	12
160	13 07.0	-34 12	8.5	10.4	<i>sp</i>	31	.396	140	4.20	91
161	15.6	-36 07	12.0	13.2	<i>sf</i>	79	.048	277	6.07	03
162	24.1	-31 50	11.3	12.5	<i>sf</i>	33	.172	264	5.34	07
163	30.0	-31 44	12.8	15.1	<i>sp</i>	7	.052	192	4.55	09
164	14 07.7	-33 27	14.8	14.9	<i>sp</i>	6	.173	264	4.07	24
165	09.3	-31 37	14.2	14.2	<i>f</i>	5	.061	262	4.34	11
166	34.1	-31 50	13.4	13.6	<i>nf</i>	134	.047	170	6.52	02
167	40.5	-37 12	11.2	14.2	<i>sf</i>	11	.106	76	4.38	21
168	42.0	-36 22	11.7	12.2	<i>sf</i>	12	.131	263	4.35	25
169	43.8	-36 59	8.5	14.0	<i>nf</i>	69	.124	191	5.24	19
170	52.5	-38 45	14.7	15.8	<i>np</i>	6	.189	190	4.06	24
171	54.0	-32 00	11.5	13.8	<i>sp</i>	15	.093	262	4.67	15
172	15 17.1	-35 30	13.5	14.6	<i>nf</i>	105	.136	233	5.95	06
173	38.6	-38 00	14.2	14.3	<i>f</i>	10	.130	165	4.47	16
174	16 13.5	-37 18	12.0	16.0	<i>sp</i>	7	1.215	325	3.11	250
175	51.8	-39 58	12.7	13.5	<i>nf</i>	25	.169	232	4.83	17
176	17 00.1	-32 42	12.5	14.4	<i>nf</i>	71	.074	211	5.87	04
177	10.8	-32 22	10.6	15.9	<i>sp</i>	34	.048	87	5.41	06
178	12.1	-34 53	7.4	10.0	<i>sf</i>	31	1.186	100	3.68	300
179	25.2	-30 36	14.7	15.3	<i>np</i>	8	.267	226	4.08	31
180	39.6	-32 02	11.8	12.6	<i>p</i>	21	.271	194	4.42	37
181	44.3	-33 11	11.6	14.9	<i>s</i>	105	.070	188	6.08	04
182	20 01.4	-32 01	13.0	13.0	<i>sf</i>	41	.786	158	4.39	78
183	19.2	-36 36	13.0	13.0	<i>sf</i>	11	.089	222	4.57	14
184	29.9	-33 22	14.7	15.6	<i>np</i>	91	.103	147	6.09	03
185	56.1	-31 52	14.4	15.0	<i>s</i>	59	.172	188	5.54	08
186	21 10.0	-39 10	13.3	14.6	<i>sf</i>	7	.096	101	4.32	16
187	12.1	-29 56	9.3	14.7	<i>s</i>	27	.151	172	4.63	30
188	16.9	-38 26	12.9	14.5	<i>np</i>	8	.147	225	4.09	30
189	42.2	-38 17	12.0	13.0	<i>nf</i>	23	.150	129	4.76	19
190	48.9	-36 27	13.1	13.7	<i>np</i>	59	.092	120	5.68	06
191	53.2	-36 27	12.8	13.6	<i>np</i>	96	.126	97	5.84	06
192	22 21.7	-34 42	14.5	14.5	<i>nf</i>	8	.187	99	4.19	24
193	39.4	-33 46	13.5	14.8	<i>sf</i>	36	.213	116	5.06	15

Phoenix, Ariz. :

1933 November 1.

PROPER MOTIONS OF FAINT STARS IN A REGION CENTRED ON α BOOTIS.

W. J. Luyten.

In addition to the proper motion survey of the southern hemisphere on Bruce plates, a few pairs of plates covering special regions in the northern hemisphere have also been investigated—mostly for the purpose of searching for companions to bright stars. Since the few stars with large proper motion found in this way do not statistically belong to the large number found in the general survey, it seems advisable to publish them separately. The first region blinked is centred on α Bootis; the plates, which were kindly lent by Dr. Shapley, have a scale value of 1 mm. = $59''.3$, an interval of 33.8 years, and show stars down to magnitude 16.5. In addition to α Bootis, which is too bright to be measured on such plates, 44 other stars were found with an appreciable proper motion, including 3 stars with a motion of more than $0''.5$ annually, and one double star. The data for them are given in the table below, where the magnitudes are on the international photographic scale. The mean error (internal) in each component of proper motion has been found to be approximately $0''.023$. It is seen that the proportion of stars belonging to the two drifts is roughly as 2 : 1.

	Mag.	R.A. 1900 14 ^h m	Dec. 1900 + °	$\mu_{\alpha} \cos \delta$ (unit $''\cdot01$)	μ_{δ}		Mag.	R.A. 1900 14 ^h m	Dec. 1900 + °	$\mu_{\alpha} \cos \delta$ (unit $''\cdot01$)	μ_{δ}
1	13.3	01.5	18 10	-19	-01	26	16.0	13.9	18 39	-16	+05
2	13.5	02.0	20 03	-26	-10	27	15.0	13.9	18 39	-16	+05
3	13.5	03.0	18 04	-11	-07	28	12.8	14.0	20 05	-09	-16
4	12.8	04.0	18 00	-13	-23	29	13.6	14.0	17 51	-11	-01
5	13.2	04.8	21 36	-21	-18	30	15.3	14.1	21 18	-12	+01
6	14.6	05.2	21 22	-25	-07	31	15.5	14.2	18 44	-07	00
7	15.5	06.1	19 31	-09	-04	32	16.4	14.3	18 01	-03	+04
8	14.8	06.2	19 09	+02	-10	33	15.0	14.8	20 33	-05	+07
9	16.5	06.7	18 31	-12	-04	34	15.9	16.2	20 02	+02	-13
10	14.8	08.7	19 40	-18	-10	35	16.3	16.4	18 29	+05	-12
11	11.6	08.8	18 37	-01	-13	36	12.2	17.5	20 41	+19	-23
12	15.3	10.0	19 16	-12	+04	37	15.2	17.5	18 36	-11	-01
13	13.3	11.0	17 57	-15	00	38	14.0	18.2	18 26	-50	-07
14	15.4	11.0	19 34	-06	-10	39	14.4	18.3	20 56	-23	-03
15	15.4	11.2	19 54	-10	-05	40	11.0	18.6	19 00	-28	-15
16	14.7	11.2	20 41	-13	-03	41	14.3	19.0	21 28	-69	-02
17	12.2	11.9	20 26	-03	-12	42	15.9	20.0	18 46	-09	-01
18	15.6	12.0	18 29	+04	-15	43	6.0	20.9	21 03	+22	-60
19	15.5	12.6	21 40	+04	-07	44	15.6	21.0	19 38	+02	-17
20	15.7	12.9	21 21	+07	-17						
21	14.2	13.0	18 50	-11	-01						
22	13.2	13.3	19 31	-06	-13						
23	13.9	13.3	17 44	-11	-02						
24	12.9	13.5	18 24	+05	-09						
25	14.3	13.6	19 26	-09	+17						

* ft star, $sp\ 15''$.

Errata.

Dr. Chandrasekhar's paper :

P. 25, equation (48), *for* = *read* $\rightarrow$.

P. 31, equations (69), (70), (71) and (73) : on the left-hand side a factor
of $\frac{2\pi}{\lambda}$ has been omitted.

Rev. M. Davidson's paper :

P. 79, line 19 from top, and p. 82, line 2, *for* G.C.T. *read* G.M.T.

P. 80, line 10 from top, *for* 9.895539 *read* 0.895539.

P. 81, line 4 from foot, *for* 0.896772 *read* 0.897672.